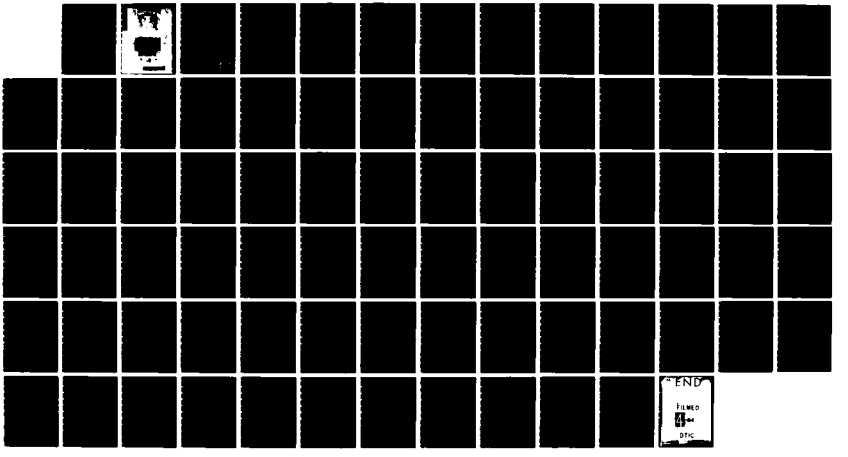
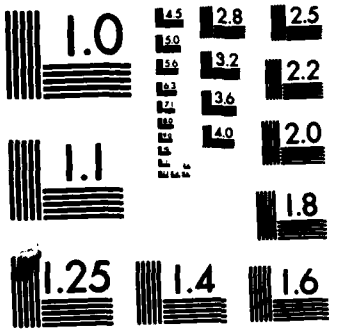
AD-A138 374 INVESTIGATION OF THE NUMERICAL METHOD OF MOMENTS FOR DIGITAL COMPUTER DET..(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH SCHOOL OF ENGI.. R E CLAPP

1/1

UNCLASSIFIED OCT 83 AFIT/GEP/PH/83D-3 F/G 12/1 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

INVESTIGATION OF THE NUMERICAL METHOD OF MOMENTS FOR DIGITAL COMPUTER DETERMINATION OF GREEN'S FUNCTIONS

THESIS

AFIT/GEP/PH/83D-3

Randolph E. Clapp
1/Lt USAF

INVESTIGATION OF THE NUMERICAL METHOD OF MOMENTS FOR DIGITAL COMPUTER DETERMINATION OF GREEN'S FUNCTIONS

THESIS

AFIT/GEP/PH/83D-3 Randolph E. Clapp
1/Lt USAF

AFIT/GEP/PH/83D-3

INVESTIGATION OF THE NUMERICAL METHOD OF MOMENTS FOR DIGITAL COMPUTER DETERMINATION OF GREEN'S FUNCTIONS

THESIS

Presented to the Faculty of the School of Engineering of
the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the Degree of
Master of Science

by

Randolph E. Clapp, B.S.
1/Lt USAF

Graduate Engineering Physics
October 1983

Preface

This report is the result of a twelve week study on the feasibility of using the method of weighted residuals to determine approximations to the discrete Green's function or an analog to it. The study was sponsored by Mr. Nick Pagano, AFWAL/MLBM. Included in this report are derivations of the methods of Galerkin, collocation, and finite differences, for the one and two-dimensional Poisson's equation. The analytical solutions for several inhomogeneity terms are also presented. The results are given in both tabular and graphical form for clarity and ease of reference. Where the results showed a significant trend or deviated from expected values, I have attempted to provide an explanation. All of the primary goals of the study were met. During the course of the twelve weeks I learned much about the theory of Green's functions and methods of numerical analysis.

I wish to acknowledge Dr. Bernard Kaplan for his support and direction. His valuable suggestions aided me to surmount difficulties which otherwise might have hindered the completion of this study.

I must also thank Dr. Hengehold, Lt Col Bailey, and Maj Cook for their assistance.

Randolph E. Clapp

Contents

List of Figures

List of Tables

List of Tables

Abstract

The purpose of this report was to determine the feasibility of using the method of weighted residuals to obtain approximations to the discrete Green's function or an analog to it. The methods of Galerkin, collocation, and finite differences were programmed on a CDC 7600 computer in Fortran IV. The resulting program was used to generate the approximate functions for the one and two-dimensional Poisson's equation. The two-dimensional case was restricted to the methods of Galerkin and finite differences on a rectangular body. The approximate Green's functions and analogs were applied to a series of inhomogeneity terms to obtain the approximate solutions. The results were compared to the analytical values at points of interest. The average percent error of the approximate solutions is reported for each case as the number of interior nodes of the mesh was increased. The areas of consideration were: the rate of convergence of the approximate solutions toward the analytical solution, the amount of computer-time required to execute the methods, and the accuracy of the approximate solutions. The results of this study indicate that the Green's functions and analogs obtained are valid approximations to the discrete Green's function itself, with the restrictions that additional calculations may be required in the case of the Galerkin approximations and excessive computer-time may occur for high-accuracy approximations. The finite difference approximations were determined to be the best method to use.

I. Introduction

Background

In many areas of scientific research one of the major steps in the theoretical modeling of experimental data is the solution of one or more linear differential equations. Often these are complex, containing many derivatives and source terms. Such equations are difficult to solve analytically. In addition, it is sometimes necessary to solve large sets of similar equations to determine trends and regions of validity for theoretical predictions. It is in these areas where computer-aided approximations to the differential equations are useful, for they can yield near-correct answers with only a small amount of computer-time.

Perhaps the most versatile of these approximations is the Green's function. It allows the reduction of a set of linear differential equations to a set of algebraic equations involving integrals. The solution set is then found by solving the resulting matrix equation. The exact analytical expression for the Green's function is often very difficult to determine.

In contrast, the discrete Green's function, applicable only at specific points, can be approximated with much less effort by using finite difference methods. The solution set can then be obtained to the desired accuracy at these discrete points on the body under examination.

Purpose

The purpose of this study is to determine the feasi-

bility of using the method of weighted residuals (MWR) to obtain the discrete Green's function or a function analogous to it. The methods of Galerkin and collocation will be studied in comparison with the method of finite differences. Both one and two-dimensional problems will be examined.

Plan of Attack

The approach to this study will be as follows:

(1) To develop a computer program which uses the methods of Galerkin, collocation, and finite differences to obtain approximations for both the discrete Green's function, or its analog, and the solution for the one-dimensional Poisson's equation. Homogeneous Dirichlet boundary conditions will be assumed in all cases.

(2) To analyze the usefulness of the Green's functions or analogs obtained in the previous step when the inhomogeneity term is varied. Areas of consideration will be: the number of calculations required, computer analysis time, convergence rate, and overall solution accuracy.

(3) To adapt the one-dimensional program to handle two-dimensional problems on a rectangular body. Only the methods of Galerkin and finite differences will be examined. Homogeneous Dirichlet boundary conditions will again be assumed.

(4) To examine the usefulness of the Green's functions or analogs obtained as in the above step.

(5) To determine the feasibility and possible directions of continued research into this approximation method.

II. Poisson's Equation in One Dimension

The first problem to be examined in this study is the one-dimensional Poisson's equation. The general form of the equation to be solved can be expressed as

$$L\mathrm{T}(x) = g(x) \tag{1}$$

where

L = The linear differential operator, $\frac{d^2}{dx^2}$

T(x) = The desired solution function

g(x) = The inhomogeneity term

with the associated homogeneous Dirichlet boundary conditions

$$T(0) = 0 \tag{2-a}$$

$$T(a) = 0 \tag{2-b}$$

Analytical Solution

The general solution to Eq(1) can be determined by direct integration. For the case of a constant inhomogeneity the general solution becomes

$$T(x) = \frac{Cx^2}{2} + C_1 x + C_2 \tag{3}$$

where C, C_1, and C_2 are constants. When the boundary conditions (Eq(2)) are applied, Eq(3) then becomes

$$T(x) = \frac{Cx^2}{2} - \frac{Cax}{2} \tag{4}$$

The analytical solution for cases with other inhomogeneity terms can be found in a similar manner.

Numerical Approximation

In all of the approximation methods discussed in this study the body under consideration is modeled by a mesh. The method is then applied at the N interior nodal points of the mesh. This results in a set of N simultaneous algebraic equations which, when solved, yields the approximate solution to the given problem. Depending upon which method is applied this approximation is valid for either the entire body or the nodal points alone.

The accuracy of the approximate solution depends on the number of nodal points in the mesh. A sample mesh for the one-dimensional problem with four interior nodes is shown in Figure 1. The nodes are numbered consecutively from left to right. The smaller the step size, h, the larger the number of nodal points and the more accurate the approximation.

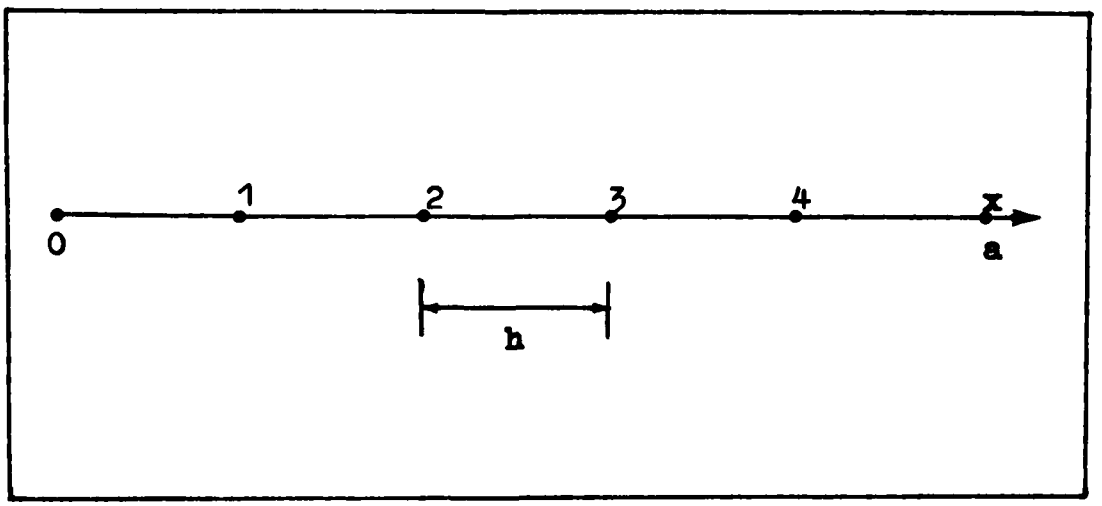


Figure 1. Sample Mesh With Four Interior Nodes

Method of Weighted Residuals. In the general method of weighted residuals (MWR) the solution is approximated as a sum of N trial functions, $f_n(x)$, weighted by undetermined factors, α_n, as (Ref 1:339)

$$T_N(x) = \sum_{n=1}^{N} \alpha_n \cdot f_n(x) \tag{5}$$

The trial functions are chosen to be linearly independent members of a complete set, each of which satisfy the given boundary conditions (Eq(2)). According to Özisik (Ref 1: 340), the trial functions for rectangular coordinates can be chosen as products of a function, w(x), with various powers of x, where w(x) is defined as

$$\begin{aligned} w(x) &= (x-0)\cdot(x-a) \\ &= x(x-a) \end{aligned} \tag{6}$$

It is obvious that Eq(6) satisfies Eq(2). For all cases of the one-dimensional problem the trial functions will be defined as

$$f_n(x) = x^n \cdot w(x) \tag{7}$$

If Eq(5) is substituted into Eq(1) we obtain (Ref 1: 339)

$$L\left[\sum_{n=1}^{N} \alpha_n \cdot f_n(x)\right] = g(x) \tag{8}$$

Multiplying Eq(8) by some weighting functions, ω_m, and then integrating over the limits of the body yields

$$\sum_{n=1}^{N} \alpha_n \int_0^a \omega_m(x) \cdot Lf_n(x)dx = \int_0^a \omega_m(x) \cdot g(x)dx \qquad (9)$$

for

$$m = 1,2,3, \ldots ,N$$

Eq(9) represents a set of N simultaneous equations with N unknowns, α_n. If we now define the inner products

$$I_{mn} = \langle \omega_m, Lf_n \rangle = \int_0^a \omega_m(x) \cdot Lf_n(x)dx \qquad (10\text{-a})$$

$$g_m = \langle \omega_m, g \rangle = \int_0^a \omega_m(x) \cdot g(x)dx \qquad (10\text{-b})$$

Eq(9) becomes

$$\sum_{n=1}^{N} \alpha_n \langle \omega_m, Lf_n \rangle = \langle \omega_m, g \rangle \qquad (11)$$

or in matrix form (Ref 2:6)

$$\bar{\bar{I}}\,\bar{\alpha} = \bar{g} \qquad (12)$$

where

= denotes a square matrix

- denotes a column vector

$$\bar{\bar{I}} = \begin{bmatrix} \langle \omega_1, Lf_1 \rangle & \langle \omega_1, Lf_2 \rangle & \cdots & \langle \omega_1, Lf_N \rangle \\ \langle \omega_2, Lf_1 \rangle & \langle \omega_2, Lf_2 \rangle & \cdots & \langle \omega_2, Lf_N \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \omega_N, Lf_1 \rangle & \langle \omega_N, Lf_2 \rangle & \cdots & \langle \omega_N, Lf_N \rangle \end{bmatrix} \tag{13}$$

$$\bar{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_N \end{bmatrix} \tag{14}$$

$$\bar{g} = \begin{bmatrix} \langle \omega_1, g \rangle \\ \langle \omega_2, g \rangle \\ \vdots \\ \langle \omega_N, g \rangle \end{bmatrix} \tag{15}$$

The weighting factors can then be determined by

$$\bar{\alpha} = \bar{\bar{I}}^{-1} \bar{g} \tag{16}$$

Finally, the approximate solution can be written as

$$T_N(x) = \tilde{f}\,\bar{\alpha} \tag{17}$$

for a method valid over the entire body, or

$$\bar{T}_N = \bar{\bar{f}}\,\bar{\alpha} \tag{18}$$

for a method valid only at discrete points, where

$$\tilde{f} = \left[f_1, f_2, f_3, \ldots, f_N\right] \tag{19}$$

$$\bar{T}_N = \begin{bmatrix} T_N(x_{\text{node } 1}) \\ T_N(x_{\text{node } 2}) \\ \vdots \\ T_N(x_{\text{node } N}) \end{bmatrix} \tag{20}$$

$$\tilde{\tilde{f}} = \begin{bmatrix} f_1(x_{\text{node } 1}), & f_2(x_{\text{node } 1}), & \ldots, & f_N(x_1) \\ f_1(x_{\text{node } 2}), & f_2(x_{\text{node } 2}), & \ldots, & f_N(x_2) \\ \cdot & \cdot & \cdot & \cdot \\ f_1(x_{\text{node } N}), & f_2(x_{\text{node } N}), & \ldots, & f_N(x_N) \end{bmatrix} \tag{21}$$

Green's Functions and Analogs. Given a differential equation of the form of Eq(1) and its associated boundary conditions (Eq(2)), the analytical Green's function for the problem can be determined. An in-depth analysis for the case of Dirichlet boundary conditions can be found in Stackgold (Ref 3:1-30). Once the Green's function has been determined the solution to Eq(1) with various inhomogeneity terms $g_1, g_2, g_3, \ldots, g_i$ can be found by calculating the integral

$$T_i(x) = \int_0^a G(x|x') \cdot g_i(x) dx' \tag{22}$$

where

$G(x|x')$ = The Green's function for Eq(1) with boundary conditions as in Eq(2)

x = The field point

x' = The source point

For the discrete Green's function Eq(22) takes the matrix form

$$\tilde{\bar{T}}_{N,i} = \bar{\bar{G}}_N \, \bar{g}_{N,i}(x) \, h_N \tag{23}$$

where

$\tilde{\bar{T}}_{N,i}$ = Eq(20) for the inhomogeneity $g_i(x)$

$G_{N,jk} = G_N(x_j | x_k)$

x_j = The x coordinate of the jth field node ($j = 1,2,3, \ldots,N$)

x_k = The x coordinate of the kth source node ($k = 1,2,3, \ldots,N$)

$g_{N,i}(x)$ = A column vector of order N whose elements are all equal to $g_i(x)$

h_N = The step size for N interior nodes

The tilde has been placed over the left-hand side of Eq(23) to emphasize that the discrete Green's function solution, $\tilde{\bar{T}}_N$, is not necessarily equal to the weighted residuals solution, $\bar{T}_N$.

If Eq(16) is substituted into both Eq(17) and Eq(18) we obtain the respective equations

$$T_N(x) = \tilde{\bar{f}} \, \bar{\bar{I}}^{-1} \, \bar{g} \tag{24-a}$$

and

$$\bar{T}_N = \tilde{\bar{f}} \, \bar{\bar{I}}^{-1} \, \bar{g} \tag{24-b}$$

In comparing Eq(23) and Eq(24) it should be noted that the weighted residual approximations do not contain the factor h_N. This is because the weighted residual solutions are N-term sumations whereas the discrete Green's function solution is the approximation of an integral (Eq(22)). We can identify the analogs to the discrete Green's function in Eq(24) as

$$\bar{G}_N^{\dagger} = \tilde{\bar{f}}\, \bar{\bar{I}}^{-1} \tag{25-a}$$

for a method valid over the entire body, and

$$\bar{\bar{G}}_N^{\dagger} = \tilde{\bar{\bar{f}}}\, \bar{\bar{I}}^{-1} \tag{25-b}$$

for a method valid at only discrete points, where the notation ($\dagger$) is used to denote the analogs. The functions in Eq(25) are termed analogs since the elements of the matrix $\bar{g}$, defined by Eq(10-b), in Eq(24) are not necessarily equal to g(x).

Method of Galerkin. In the method of Galerkin the weighting functions, ω_m, are taken to be equal to the trial functions, f_n, as defined in Eq(7). For these weighting functions the inner products in Eq(10) take the form

$$I_{mn} = \langle f_m, Lf_n \rangle = \int_0^a f_m(x) \cdot Lf_n(x)dx \tag{26-a}$$

$$g_m = \langle f_m, g \rangle = \int_0^a f_m(x) \cdot g(x)dx \tag{26-b}$$

The resulting form of Eq(11) is then

$$\sum_{n=1}^{N} \alpha_n \langle f_m, Lf_n \rangle = \langle f_m, g \rangle \tag{27}$$

Since there are no restrictions on the value of x in Eq(27) the method of Galerkin approximations are valid over the entire body and are given by Eq(24-a).

The appropriate form of the analog to the discrete Green's function is that of Eq(25-a). It is clear that this must be an analog since the elements of the inhomogeneity matrix, $\bar{g}$, defined by Eq(26-b), are never all equal to g(x). This analog should be as useful as the discrete Green's function itself with the exception that the inner products to determine the elements of the inhomogeneity matrix must be recalculated for each new inhomogeneity term.

Method of Collocation. In the method of collocation, or point matching, the weighting functions, ω_m, are defined as (Ref 2:10)

$$\omega_m = \delta(x-x_m) \tag{28}$$

where (x) is the Dirac delta function whose properties are given as (Ref 3:1-30)

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \tag{29-a}$$

$$\int_0^a \delta(x)dx = \begin{cases} 0 & 0 \text{ not in } (a,b) \\ 1 & 0 \text{ in } (a,b) \end{cases} \tag{29-b}$$

$$\int_{-\infty}^{\infty} \delta(x-x')f(x)dx = f(x') \qquad (29\text{-}c)$$

For these weighting functions the inner products in Eq(10) take the form

$$I_{mn} = \langle \delta(x-x_m), Lf_n \rangle = \int_0^a \delta(x-x_m) Lf_n(x)dx \qquad (30\text{-}a)$$

$$= Lf_n(x_m)$$

$$g_m = \langle \delta(x-x_m), g \rangle = \int_0^a \delta(x-x_m) g(x)dx \qquad (30\text{-}b)$$

$$= g(x_m)$$

where (Ref 2:10)

$$x_m = \frac{m\,a}{N+1} \qquad (31)$$

The resulting form of Eq(11) is then

$$\sum_{n=1}^{N} \alpha_n Lf_n(x_m) = g(x_m) \qquad (32)$$

The value of x in Eq(32) is restricted to that of the collocation points, x_m. Therefore the collocation approximations are valid only at these points and are given by Eq(24-b).

The appropriate form of the analog to the discrete Green's function is that of Eq(25-b). The elements of the

inhomogeneity matrix, $\bar{g}$, are the same as the elements of the corresponding matrix for the discrete Green's function as in Eq(23), but the approximate solutions, $\bar{T}_N$ and $\bar{\tilde{T}}_N$, are not necessarily the same since $\bar{\bar{G}}_N$ is not necessarily the same as the analog form, $\bar{\bar{\tilde{f}}}\,\bar{\bar{I}}^{-1}$. This analog should be as useful as the discrete Green's function itself except that the number of terms necessary to achieve a given level of accuracy may differ between the two methods.

Method of Finite Differences. According to Stackgold (Ref 3:1-30) the defining relation for the Green's function for the Laplacian operator is

$$\frac{d^2G(x|x')}{dx^2} = \delta(x-x') \tag{33}$$

with the associated boundary conditions for Eq(1) and Eq(2)

$$G(0|x') = 0 \tag{34-a}$$

$$G(a|x') = 0 \tag{34-b}$$

For the discrete Green's function on a mesh with step size h_N Eq(33) takes the form (Ref 4:315)

$$\frac{d^2G_N(x|x')}{dx^2} = h_N^{-1}\delta(x-x') \tag{35}$$

The derivative term in Eq(35) can be replaced by a central difference quotient as (Ref 5:6)

$$\frac{d^2G_N(x|x')}{dx^2} = \frac{G_N(x+h_N|x')-2G_N(x|x')+G_N(x-h_N|x')}{h_N^2} \quad (36)$$

Substituting Eq(36) into Eq(35) and multiplying by h_N^2 yields

$$G_N(x+h_N|x')-2G_N(x|x')+G_N(x-h_N|x') = h_N\delta(x-x') \quad (37)$$

Applying Eq(37) together with the boundary conditions (Eq(34)) to each of the N interior nodes of the mesh results in a set of N^2 simultaneous equations expressed in matrix form as

$$\bar{\bar{C}}\ \bar{\bar{G}}_N = h_N\ \bar{\bar{I}}_N \quad (38)$$

where

$\bar{\bar{C}}$ = The coefficient matrix

$$\bar{\bar{G}}_N = \begin{bmatrix} G_N(x_1|x_1) & G_N(x_1|x_2) & \cdots & G_N(x_1|x_N) \\ G_N(x_2|x_1) & G_N(x_2|x_2) & \cdots & G_N(x_2|x_N) \\ \cdots & \cdots & \cdots & \cdots \\ G_N(x_N|x_1) & G_N(x_N|x_2) & \cdots & G_N(x_N|x_N) \end{bmatrix} \quad (39)$$

$\bar{\bar{I}}_N$ = The identity matrix of order N

The solution of Eq(38) yields the approximate discrete Green's function matrix (Eq(39)). The solution to the one-dimensional Poisson's equation then follows as in Eq(23).

Computer Analysis

The numerical approximations discussed in the previous sections (Eqs(24-a),(24-b), and (23)) were programmed in For-

tran IV on a CDC 7600 mainframe. The resulting computer program is on file with the AFIT Department of Physics.

Problem Set. A series of four problems were analyzed as the number of interior nodes of the mesh was increased. The areas of consideration were: convergence rate, computer-time, and the overall solution accuracy. The four problems analyzed were:

$$\text{I)} \quad L[T(x)] = 10 \tag{40-a}$$

$$\text{II)} \quad L[T(x)] = x^2 \tag{40-b}$$

$$\text{III)} \quad L[T(x)] = x^2 + 1 \tag{40-c}$$

$$\text{IV)} \quad L[T(x)] = x^2 + x + 1 \tag{40-d}$$

The Dirichlet boundary conditions for all four problems were taken as

$$T(0) = 0 \tag{41-a}$$

$$T(2) = 0 \tag{41-b}$$

The analysis was performed at $x = 0.\overline{66}$ and $x = 1.\overline{33}$ for all cases.

Exact Solution. The analytical solutions to Eqs(40) were found by direct integration to be

$$\text{I)} \quad T(x) = 5x^2 - 5x \tag{42-a}$$

$$\text{II)} \quad T(x) = \frac{x^4}{12} - \frac{16x}{24} \tag{42-b}$$

$$\text{III)} \quad T(x) = \frac{x^4}{12} + \frac{x^2}{2} - \frac{40x}{24} \tag{42-c}$$

$$\text{IV)} \quad T(x) = \frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2}{2} - \frac{56x}{24} \tag{42-d}$$

The values of the above solutions at the points of interest are listed in Table I.

Average Error. The average percent error for the one-dimensional problem is defined as

$$\langle E_N \rangle = \frac{100}{2}\left|\frac{|T_N(0.\overline{66})-T(0.\overline{66})|}{T(0.\overline{66})} + \frac{|T_N(1.\overline{33})-T(1.\overline{33})|}{T(1.\overline{33})}\right| \quad (43)$$

where

T_N = The approximate solution under consideration

Convergence Rate. The rate at which each of the three approximate solutions (Eqs(17),(18), and (23)) converged toward the exact values listed in Table I was examined for problems I-IV. The average percent error in the approximate solutions is plotted vs the number of interior nodes in the mesh as Figures 2-5. Appendix A contains the actual values of the approximations at the points of interest for each of the four problems.

Table I

Exact Solution Values at Points of Interest

Problem No.	Point 1 x=0.$\overline{66}$	Point 2 x=1.$\overline{33}$
I	-4.$\overline{44}$	-4.$\overline{44}$
II	-0.42798	-0.62551
III	-0.87243	-1.06996
IV	-1.26749	-1.56379

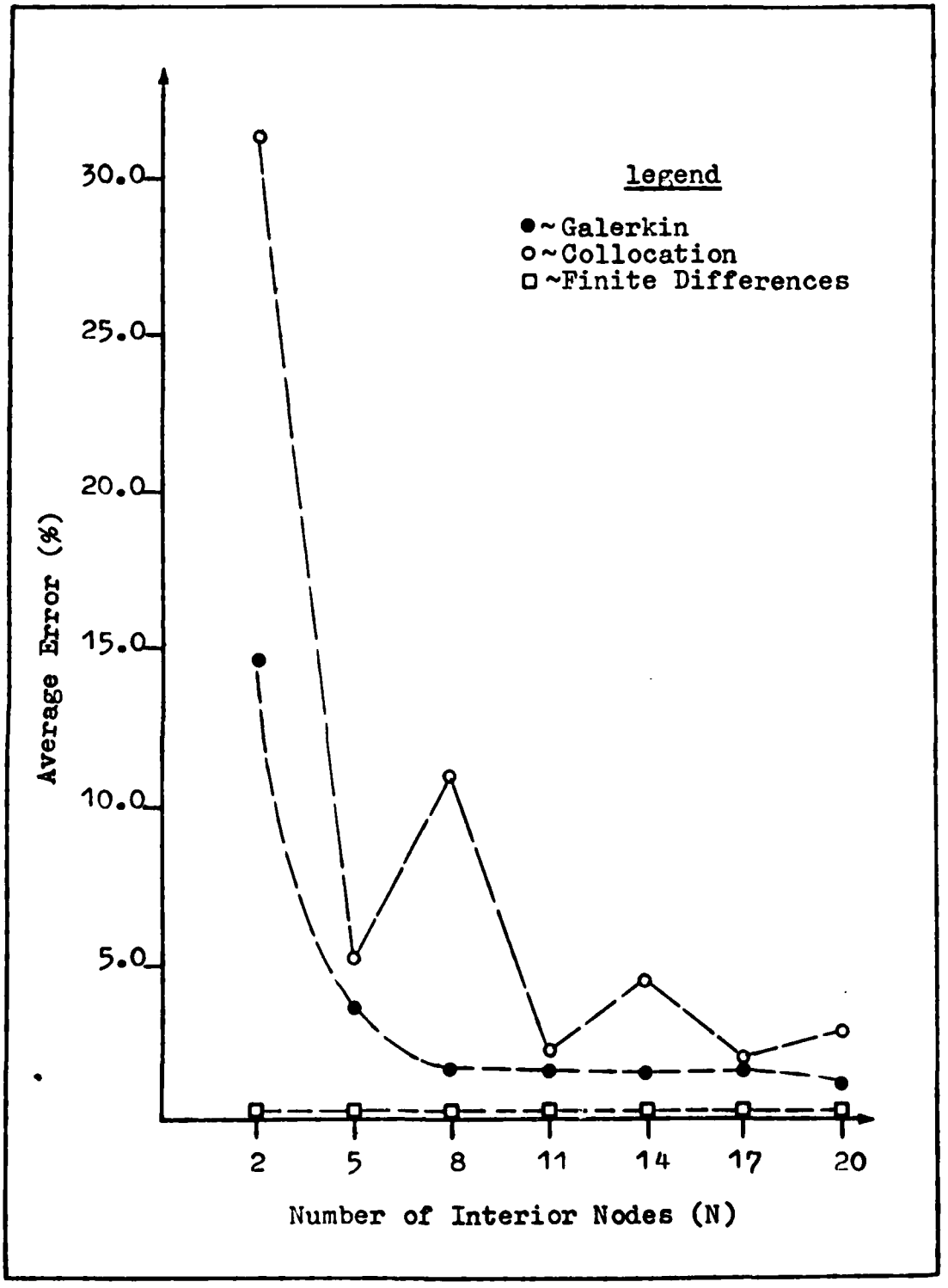


Figure 2. Average Error vs Node Number for g(x) = 10

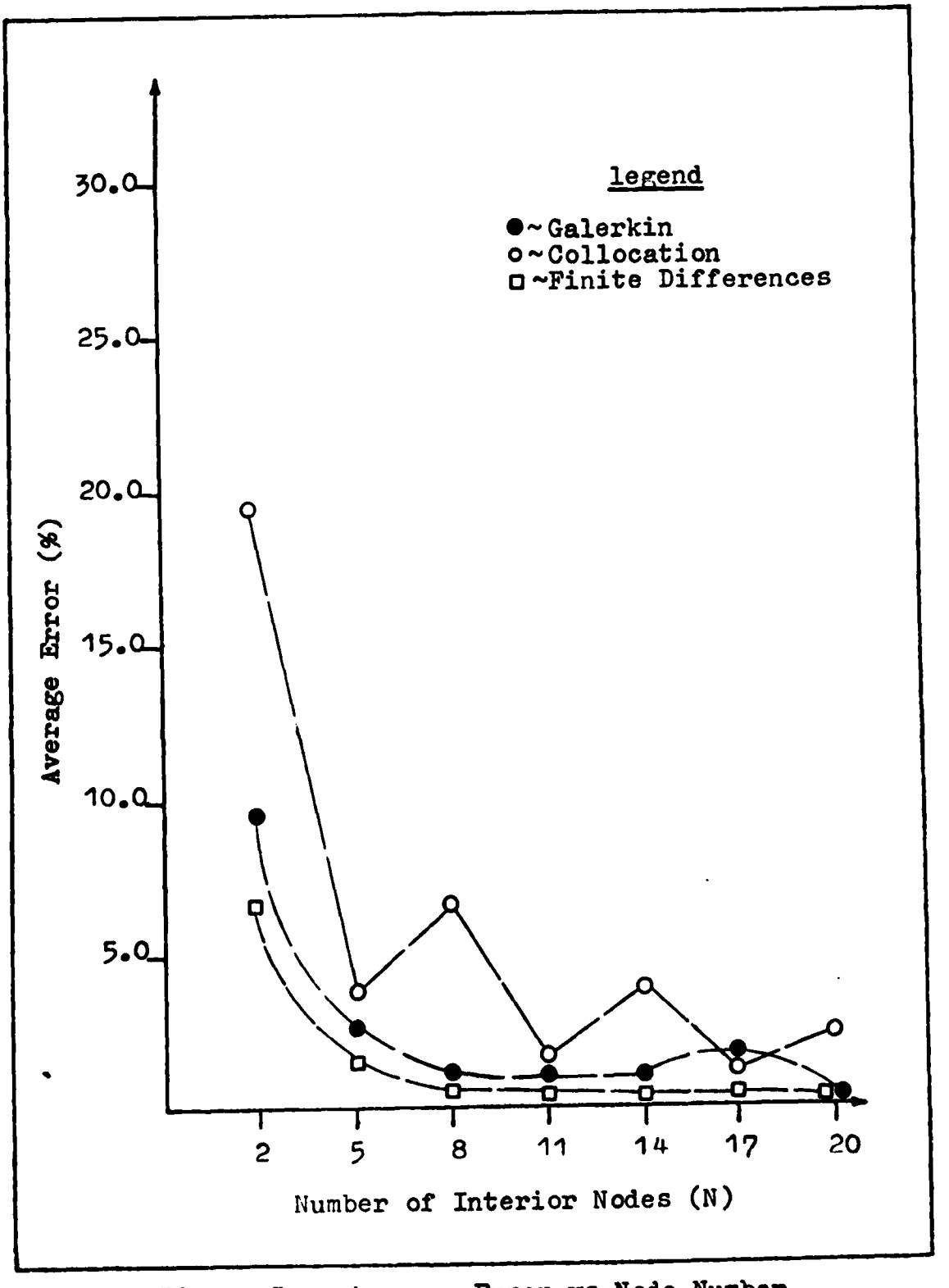


Figure 3. Average Error vs Node Number for $g(x) = x^2$

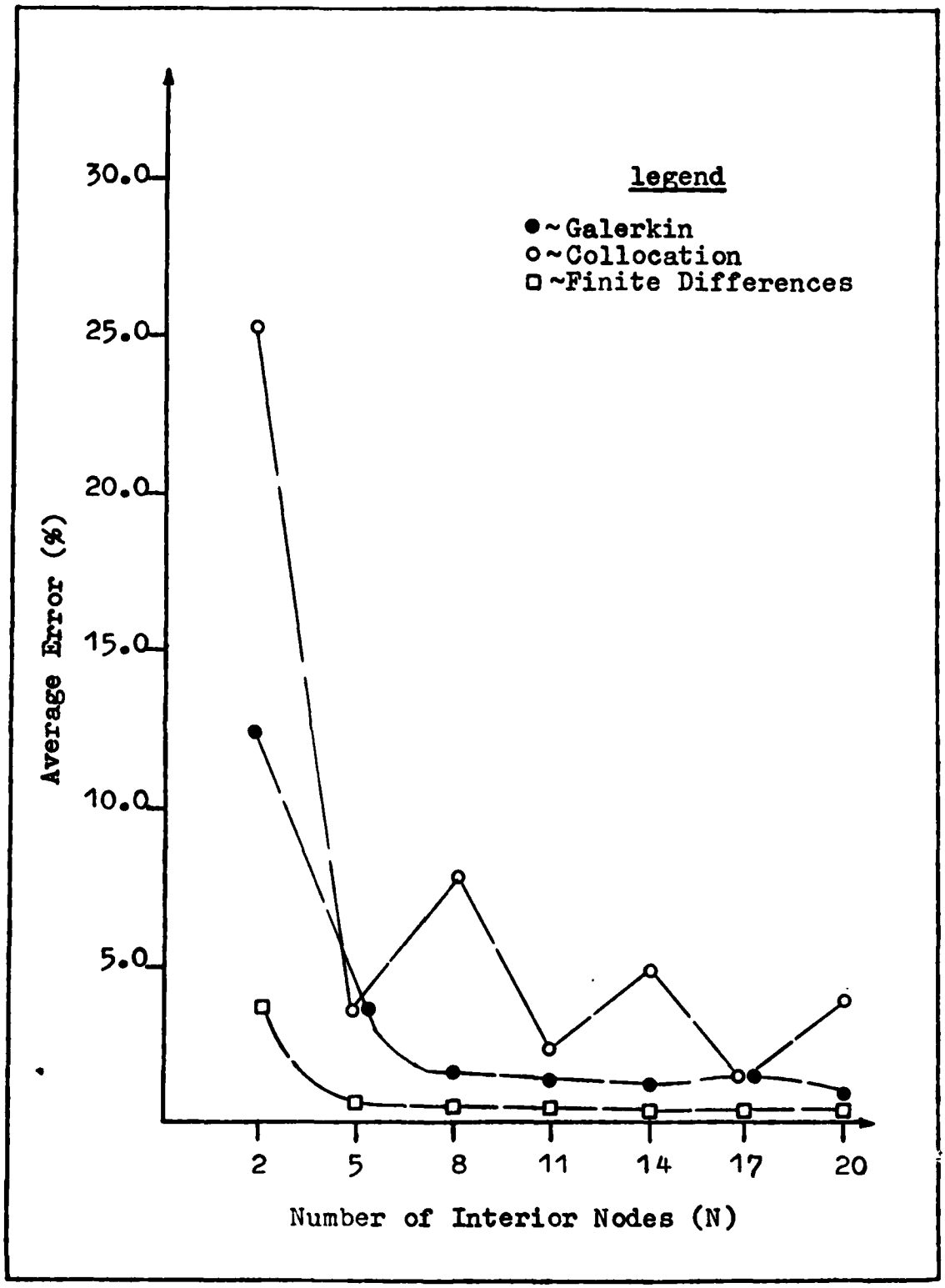


Figure 4. Average Error vs Node Number for $g(x) = x^2 + 1$

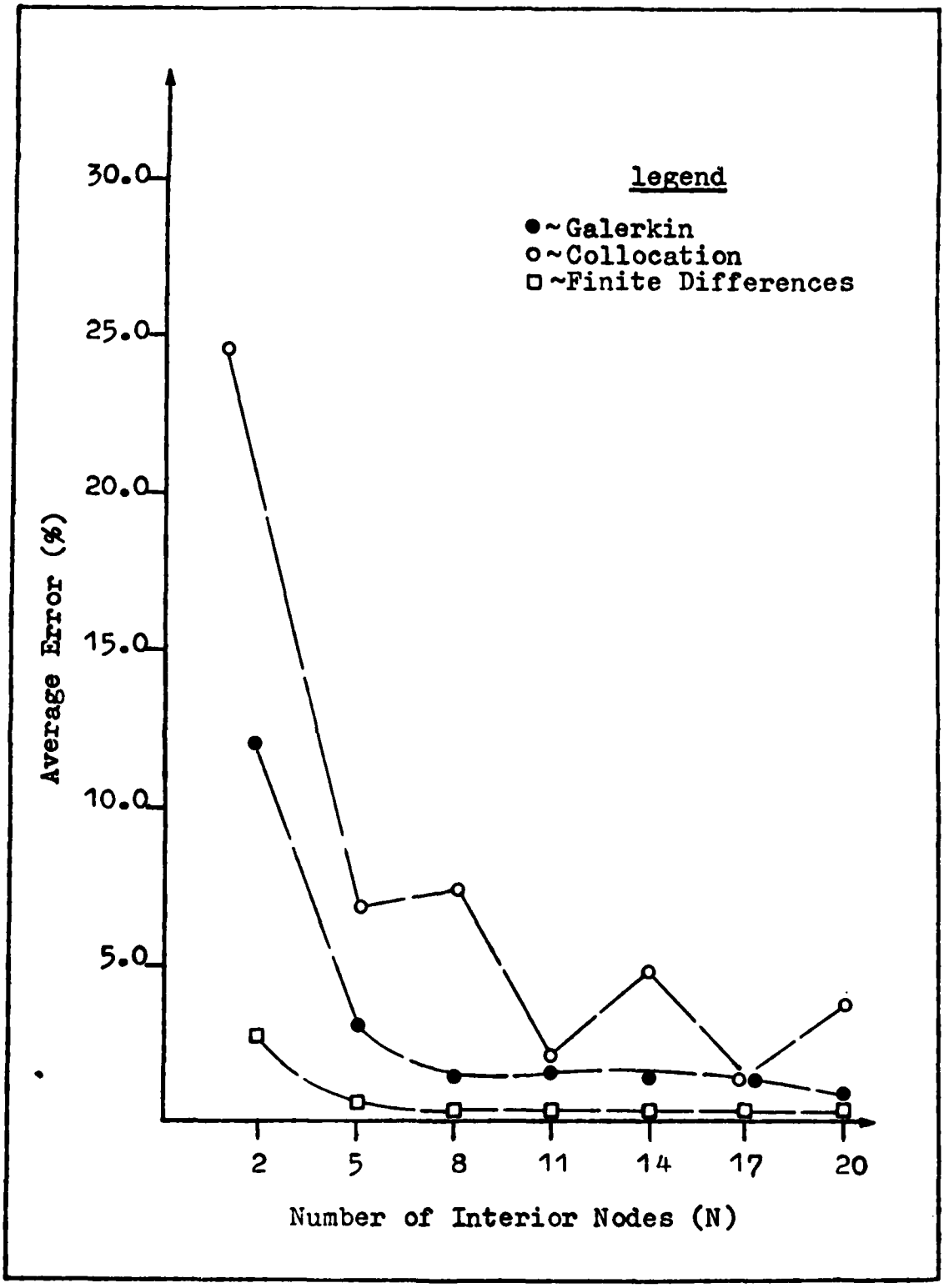


Figure 5. Average Error vs Node Number for $g(x) = x^2 + x + 1$

In each of the four cases the finite difference approximation yielded the best initial solution. In the case of problem I the method yielded the exact values. Average errors below one percent for eight or more interior nodes were achieved.

The method of Galerkin approximations converged at a somewhat slower rate and did not achieve an average error below one percent until 20 interior nodes were used. In fact a slight increase in the error was seen at 17 nodes in each case examined. This was followed by a decrease to below one percent at 20 nodes.

The collocation approximations yielded the worst initial solutions and showed marked oscillations in the average percent error. The oscillations were damped such that the error still dropped to around one percent at the 17 node point. The slight increase and subsequent decrease in the Galerkin approximations mentioned above may be evidence of a smaller amplitude oscillation there also.

The method of weighted residuals is sensitive to the choice of the trial functions. It may be that the oscillations in the collocation and Galerkin approximations are a result of an inappropriate choice of trial functions for the analysis of Poisson's equation. Babuska (Ref 6:241-245) states that the trial functions as given in Eq(7) are numerically unstable for Gaussian elimination matrix inversion. That is, the matrix $\overline{\overline{I}}$ in the weighted residual solutions is very nearly singular creating significant errors in the cal-

culated inverse. Although Gaussian elimination was not used in the direct matrix inversion routine it is reasonable to expect that the same problem may be present in many inversion methods. A different choice of trial functions may lessen or eliminate the problem.

Computer Analysis Time. The quantity used as a measure of the computer analysis time was the actual execution time of the program. The time taken by compilation, input, and output was almost constant for all computer runs. The execution time in seconds is plotted in Figures 6-9 vs the number of interior nodes in the mesh.

For all cases the execution time was the same for each approximation method at eight or less nodes. Above eight nodes the execution time steadily increased. This was to be expected since an increased number of nodes requires additional iterations of loops in the program.

The method of finite differences required the least amount of computer time (less than 0.2 seconds in all cases) with the weighted residual methods requiring considerably more. It should be noted that for problems III and IV the Galerkin method required less time than it did for problems I and II. This could not be explained. The increased time required by the weighted residual methods can be explained as the result of the additional matrix multiplication required.

Overall Solution Accuracy. All three approximation methods yielded solutions with an average percent error of

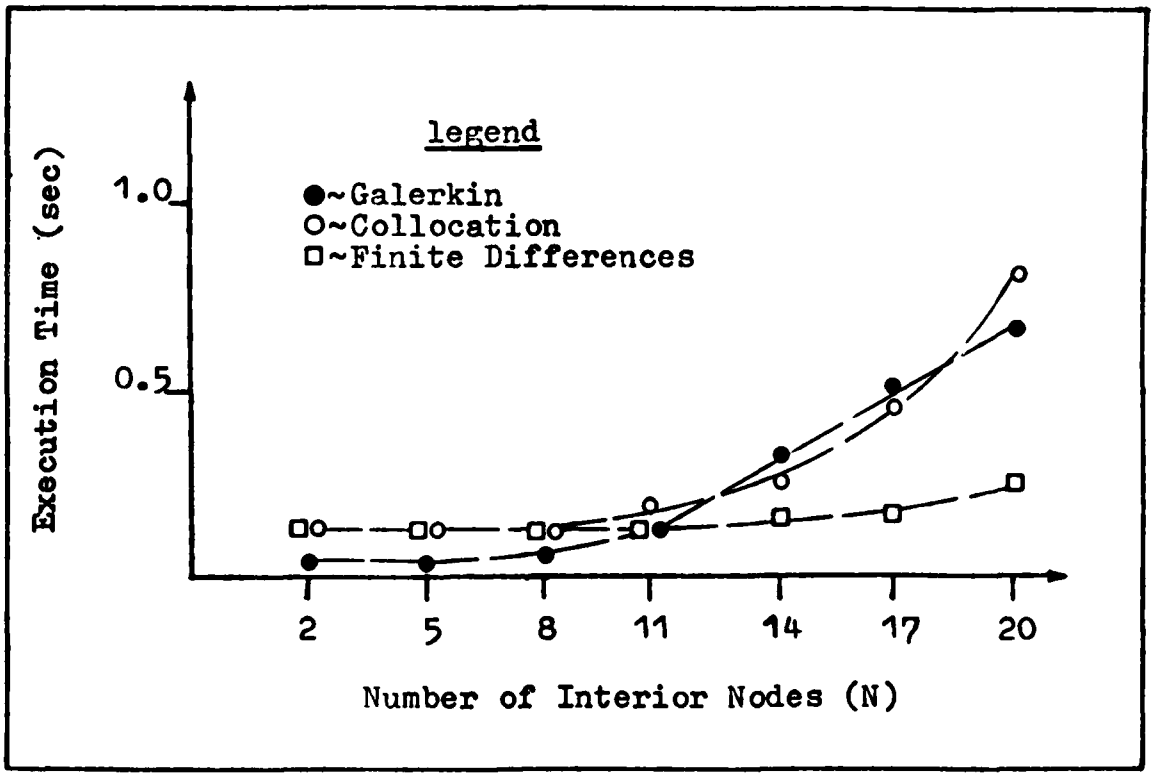


Figure 6. Execution Time vs Node Number for $g(x) = 10$

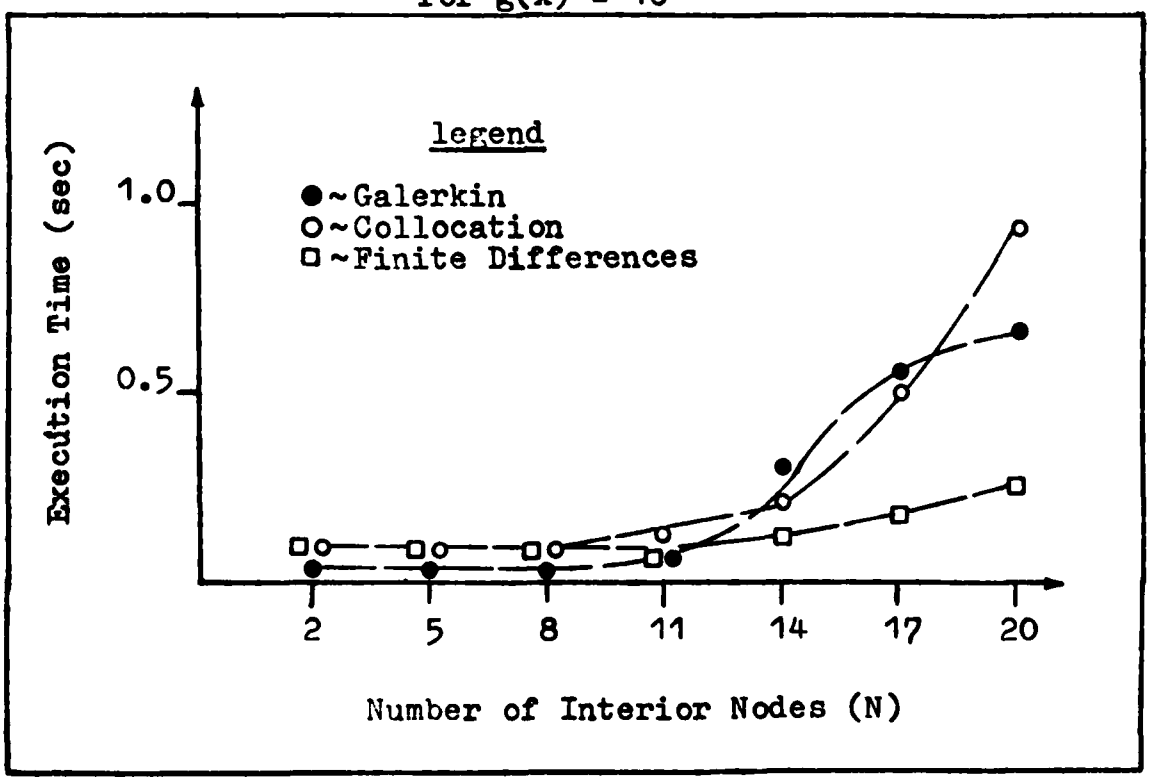


Figure 7. Execution Time vs Node Number for $g(x) = x^2$

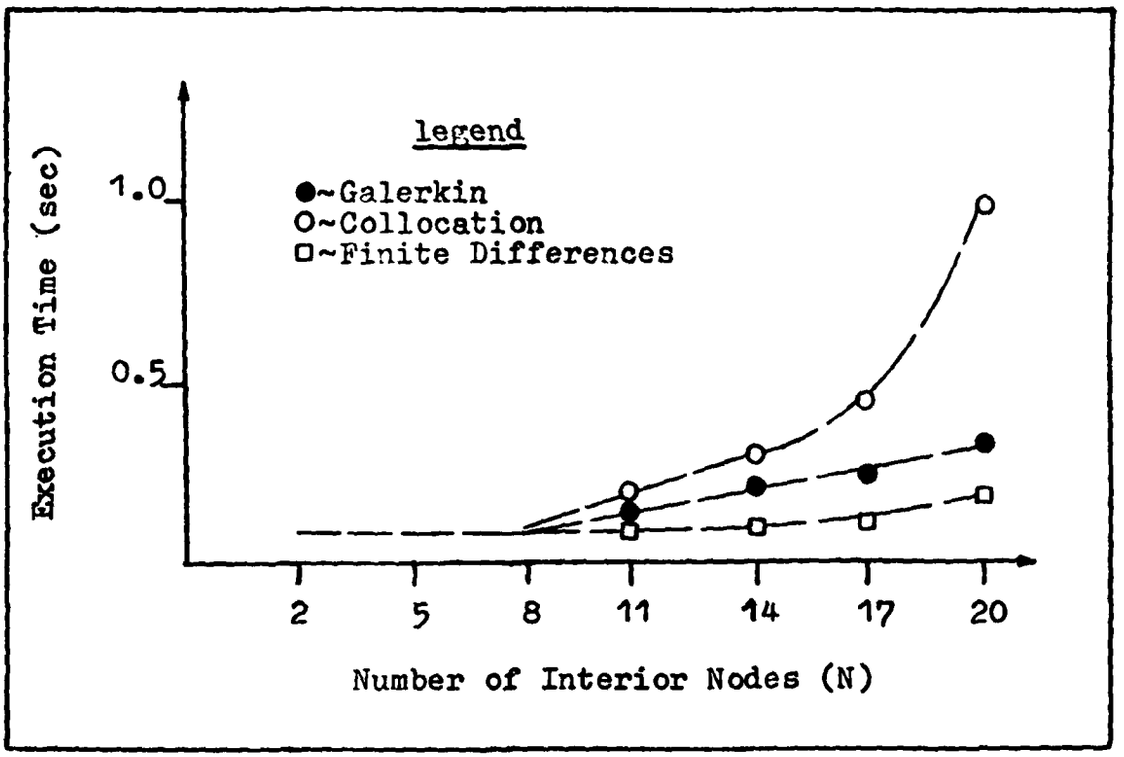


Figure 8. Execution Time vs Node Number for $g(x) = x^2 + 1$

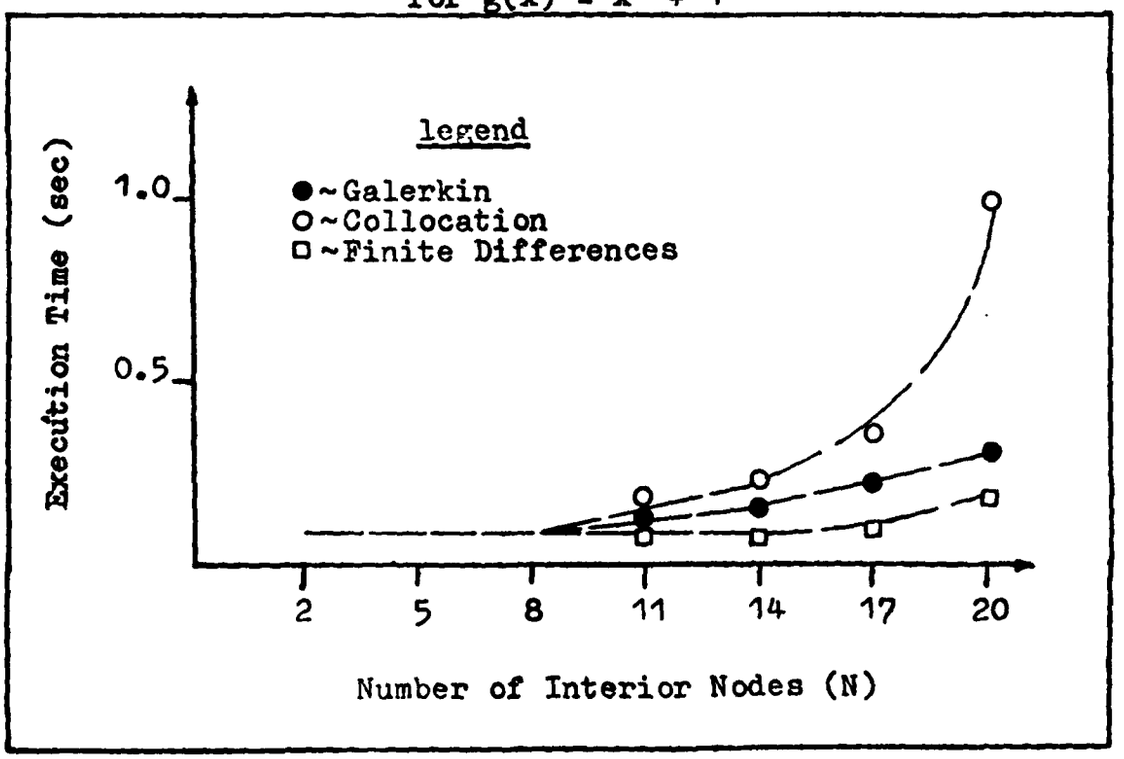


Figure 9. Execution Time vs Node Number for $g(x) = x^2 + x + 1$

about one percent when 17 or more nodes were used (taking the minimum point of the collocation oscillations). The convergence trends indicate that for more than 25 nodes the three methods should yield nearly the exact values with average errors less than 0.1 percent.

It should be noted here that for the collocation approximation the direct matrix inversion routine used (IMSL routine LINV2F; Ref 7) issued a warning that its accuracy test had failed. This occured at five or more interior nodes. Some of the oscillation in these approximations could be a result of this round-off error.

Conclusions

From the analyses performed it seems that for approximations with few terms the method of finite differences is the best choice. Not only is it easier to apply (less calculations required and less execution time) but the approximate solutions are more accurate.

For approximations of more than 10 terms the trade-offs between calculations required and approximation accuracy make the Galerkin and collocation methods about equal. The finite difference method is better than both.

III. Poisson's Equation in Two Dimensions

The second problem to be examined in this study is the two-dimensional Poisson's equation. The general form of the equation to be solved can be expressed as

$$L'T'(x,y) = g'(x,y) \tag{44}$$

where

' denotes a two-dimensional function

L' = The linear differential operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$T'(x,y)$ = The desired solution function

$g'(x,y)$ = The inhomogeneity term

with the associated homogeneous Dirichlet boundary conditions

$$T'(0,y) = 0 \tag{45-a}$$

$$T'(a,y) = 0 \tag{45-b}$$

$$T'(x,0) = 0 \tag{45-c}$$

$$T'(x,b) = 0 \tag{45-d}$$

This is Poisson's equation on a rectangular body.

Analytical Solution

The analytical solution to Eq(44) with boundary conditions Eq(45) can be found by a Fourier series expansion as (Ref 8:41-42)

$$T'(x,y) = \frac{4}{\pi^2 ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\int_0^b d\xi \int_0^a d\eta \, \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \right.$$

$$\left. \sin\left(\frac{m\pi\xi}{a}\right) \sin\left(\frac{n\pi\eta}{b}\right) g'(\xi,\eta) \Big/ \left| \left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right| \right] \quad (46)$$

For a given inhomogeneity the integration can easily be performed and the series solution can then be programmed to the desired number of terms providing Eq(46) converges.

Numerical Approximation

A sample mesh and node numbering scheme is shown as Figure 10 for the rectangular body. The x and y step sizes, h and k respectively, control the number of nodal points and hence the accuracy of the approximation.

Method of Weighted Residuals. In the two-dimensional MWR the solution is approximated as a sum of N trial functions, $f_n'(x,y)$, weighted by undetermined factors, α_n', as (Ref 1:339)

$$T_N'(x,y) = \sum_{n=1}^{N} \alpha_n' \, f_n'(x,y) \quad (47)$$

The trial functions each satisfy the given boundary conditions (Eq(45)). According to Ozisik (Ref 1:340-344), the trial functions for rectangular coordinates can be chosen as products of a function, $w'(x,y)$, with various powers of x and y, where $w'(x,y)$ is defined as

$$w'(x,y) = (x-0)\bullet(y-0)\bullet(x-a)\bullet(y-b) \quad (48)$$

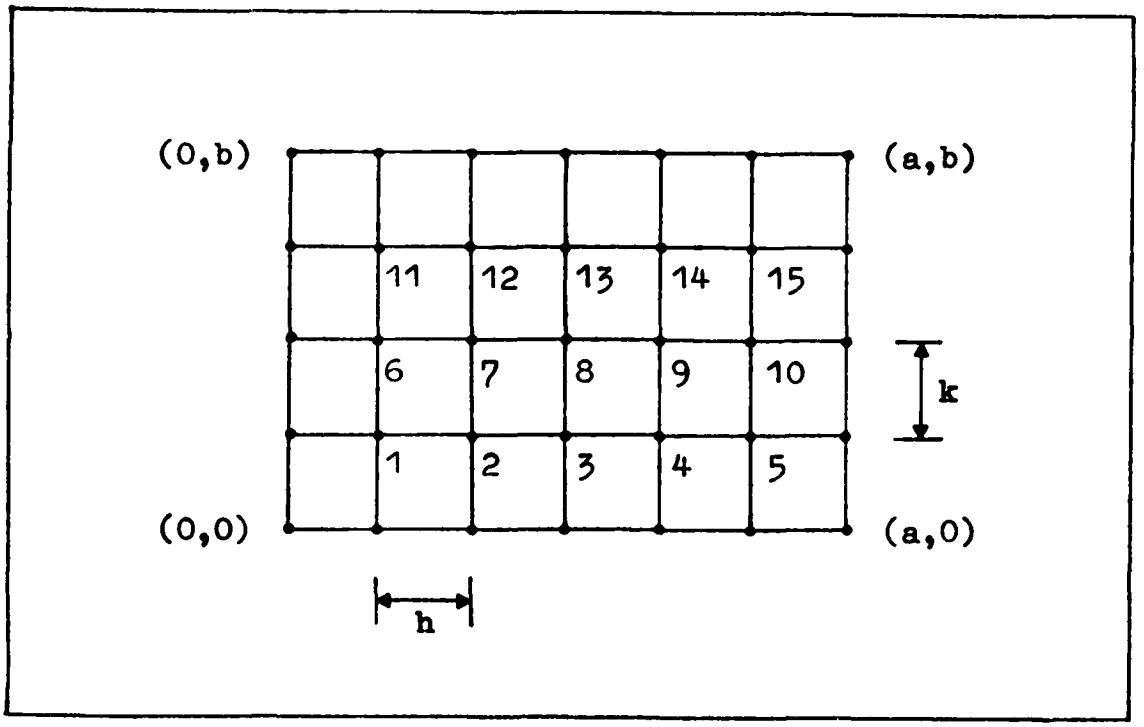


Figure 10. Sample Mesh for a Rectangular Body

Eq(48) satisfies the boundary conditions as expressed in Eq(45). For all cases of the two-dimensional problem the trial functions will be defined as

$$f_n' = (xy)^{\frac{2n-2}{3}} \cdot w'(x,y) \quad \text{(for n=1,4,7, ...)} \quad (49\text{-a})$$

$$f_n' = x^{\frac{2n+2}{3}} \cdot w'(x,y) \quad \text{(for n=2,5,8, ...)} \quad (49\text{-b})$$

$$f_n' = y^{\frac{2n}{3}} \cdot w'(x,y) \quad \text{(for n=3,6,9, ...)} \quad (49\text{-c})$$

If Eq(47) is substituted into Eq(44) we obtain (Ref 1:339)

$$L'\left[\sum_{n=1}^{N} \alpha_n' \cdot f_n'(x,y)\right] = g'(x,y) \quad (50)$$

Multiplying Eq(50) by some weighting functions, ω_m', and then integrating over the limits of the rectangle yields

$$\sum_{n=1}^{N} \alpha_n' \int_0^b \int_0^a \omega_m'(x,y)\, L' f_n'(x,y)\,dxdy = \int_0^b \int_0^a \omega_m'(x,y)\, g'(x,y)\,dxdy \tag{51}$$

for

$$m = 1,2,3, \ldots ,N$$

This again represents a set of N simultaneous equations with N unknowns, α_n'. The matrix form of Eq(51) is the same as that for the one-dimensional analysis

$$\bar{\bar{I}}'\ \bar{\alpha}' = \bar{g}' \tag{52}$$

where

$$\bar{\bar{I}}' = \begin{bmatrix} \langle \omega_1', L' f_1' \rangle & \langle \omega_1', L' f_2' \rangle & \cdots & \langle \omega_1', L' f_N' \rangle \\ \langle \omega_2', L' f_1' \rangle & \langle \omega_2', L' f_2' \rangle & \cdots & \langle \omega_2', L' f_N' \rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \omega_N', L' f_1' \rangle & \langle \omega_N', L' f_2' \rangle & \cdots & \langle \omega_N', L' f_N' \rangle \end{bmatrix} \tag{53}$$

$$\bar{\alpha}' = \begin{bmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_N' \end{bmatrix} \tag{54}$$

$$\bar{g}' = \begin{bmatrix} \langle \omega_1', g' \rangle \\ \langle \omega_2', g' \rangle \\ \vdots \\ \langle \omega_N', g' \rangle \end{bmatrix} \tag{55}$$

$$\langle \mu, \nu \rangle = \int_0^b \int_0^a \mu \cdot \nu \, dx dy \tag{56}$$

The weighting factors can then be determined by

$$\bar{\alpha}' = \bar{\bar{I}}'^{-1} \bar{g}' \tag{57}$$

Finally, the approximate solution can be written as

$$T_N'(x,y) = \tilde{f}' \bar{\alpha}' \tag{58}$$

for a method valid over the entire body, or

$$\bar{T}_N' = \bar{\tilde{f}}' \bar{\alpha}' \tag{59}$$

for a method valid at discrete points alone, where

$$\tilde{f}' = \left[f_1', f_2', f_3', \ldots, f_N' \right] \tag{60}$$

$$\bar{T}_N' = \begin{bmatrix} T_N'(x_{\text{node } 1}, y_{\text{node } 1}) \\ T_N'(x_{\text{node } 2}, y_{\text{node } 2}) \\ \vdots \\ T_N'(x_{\text{node } N}, y_{\text{node } N}) \end{bmatrix} \tag{61}$$

$$\overline{\overline{f}}' = \begin{bmatrix} f_1'(x_1,y_1) & f_2'(x_1,y_1) & \cdots & f_N'(x_1,y_1) \\ f_1'(x_2,y_2) & f_2'(x_2,y_2) & \cdots & f_N'(x_2,y_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1'(x_N,y_N) & f_2'(x_N,y_N) & \cdots & f_N'(x_N,y_N) \end{bmatrix} \quad (62)$$

Green's Functions and Analogs. Given a differential equation of the form of Eq(44) and its associated boundary conditions (Eq(45)), the analytical Green's function for the problem can be determined. An analysis can be found in a previous thesis by Gallof (Ref 8:42-43). The two-dimensional form of Eq(22) then can be written as

$$T_i'(x,y) = \int_0^b \int_0^a G'(x|x';y|y') \cdot g_i'(x,y) dx'dy' \quad (63)$$

where

$G'(x|x';y|y')$ = The Green's function for Eq(44) with boundary conditions as in Eq(45)

x and y = The field point coordinates

x' and y' = The source point coordinates

For the discrete Green's function Eq(63) takes the matrix form

$$\overline{\overline{T}}'_{N,i} = \overline{\overline{G}}'_N \cdot \overline{\overline{g}}'_{N,i}(x,y) \cdot h_R \cdot k_S \quad (64)$$

where

$\tilde{\bar{T}}'_{N,i}$ = Eq(61) for the inhomogeneity $g'_i(x,y)$

$G'_{N,jk} = G'_N(x_j|x_k;y_j|y_k)$

x_j and y_j = The x and y coordinates at the jth field node (j=1,2,3, ... ,N)

x_k and y_k = The x and y coordinates at the kth source node (k=1,2,3, ... ,N)

h_R = The step size for R interior nodes along the x-axis

k_S = The step size for S interior nodes along the y-axis

$R \cdot S = N$

The tilde has again been placed over the left-hand side of Eq(64) to emphasize that, as in the one-dimensional case (Eq(23)), the discrete Green's function solution, $\tilde{\bar{T}}'_N$, is not necessarily equal to the weighted residuals solution, $\bar{T}'_N$.

The two-dimensional forms of Eqs(24) are the approximate solutions

$$T'_N(x,y) = \tilde{\bar{f}}'\bar{\bar{I}}'^{-1}\,\bar{g}' \qquad (65\text{-a})$$

and

$$\bar{T}'_N = \tilde{\bar{f}}'\bar{\bar{I}}'^{-1}\,\bar{g}' \qquad (65\text{-b})$$

In comparing Eq(64) and Eq(65) it should be noted that the weighted residual approximations do not contain the factors

h_R and k_S. This is because the weighted residual solutions are N-term summations whereas the discrete Green's function solution is the approximation of an integral (Eq(63)). We can identify the discrete Green's function analogs in two dimensions as

$$\bar{G}_N' = \tilde{\bar{f}}'\bar{\bar{I}}'^{-1} \tag{66-a}$$

for a method valid over the entire body, and

$$\bar{\bar{G}}_N' = \tilde{\bar{\bar{f}}}'\bar{\bar{I}}'^{-1} \tag{66-b}$$

for a method valid at discrete points only. These functions are termed analogs since the elements of the matrix $\bar{g}'$ in the weighted residuals solutions are not necessarily equal to the inhomogeneity, $g'(x,y)$.

Method of Galerkin. The weighting functions, ω_m', are again taken to be equal to the trial functions, f_n', as defined in Eq(49). The resulting form of Eq(51) is

$$\sum_{n=1}^{N} \alpha_n' \int_0^b \int_0^a f_m'(x,y) L' f_n'(x,y)\,dx\,dy = \int_0^b \int_0^a f_m'(x,y) g'(x,y)\,dx\,dy \tag{67}$$

for

$$m = 1,2,3, \ldots ,N$$

There are no restrictions on the value of x or y in Eq(67). Therefore the Galerkin approximations are valid over the entire rectangle and are given by Eq(65-a).

The appropriate form of the discrete Green's function analog in two dimensions is that of Eq(66-a). This analog should be as useful as the two-dimensional discrete Green's function itself except that the double integrals must be recalculated for each new inhomogeneity matrix.

Method of Collocation. In the two-dimensional method of collocation the weighting functions, ω_m, are taken as (Ref 2:10)

$$\omega_m' = \delta(x-x_m) \cdot \delta(y-y_m) \tag{68}$$

where $\delta(x)$ is the Dirac delta function whose properties are given in Eq(29). The coordinates x_m and y_m are defined as

$$x_m = \text{The x coordinate of the mth interior node} \tag{69-a}$$

$$y_m = \text{The y coordinate of the mth interior node} \tag{69-b}$$

For these weighting functions

$$I_{mn}' = \int_0^b \int_0^a \delta(x-x_m)\,\delta(y-y_m) L' f_n'(x,y)\,dx\,dy \tag{70-a}$$

$$= L' f_n'(x_m, y_m)$$

$$g_m' = \int_0^b \int_0^a \delta(x-x_m)\,\delta(y-y_m) g'(x,y)\,dx\,dy \tag{70-b}$$

$$= g'(x_m, y_m)$$

The resulting form of Eq(51) is then

$$\sum_{n=1}^{N} \alpha_n' L' f_n'(x_m, y_m) = g'(x_m, y_m) \qquad (71)$$

The values of x and y in Eq(71) are restricted to the coordinates of the collocation points. Therefore the two-dimensional collocation approximations are valid at the collocation points alone and are given by Eq(65-b).

The appropriate form of the discrete Green's function analog is that of Eq(66-b). As in the one-dimensional case these analogs should be as useful as the discrete Green's function itself except the convergence rate may differ between the two methods.

Method of Finite Differences. The defining relation for the Green's function for the two-dimensional Laplacian operator is (Ref 3:1-30)

$$\frac{\partial^2 G'(x|x';y|y')}{\partial x^2} + \frac{\partial^2 G'(x|x';y|y')}{\partial y^2} = \delta(x-x')\delta(y-y') \qquad (72)$$

With the associated boundary conditions for Eq(44) and Eq(45)

$$G'(0|x';y|y') = 0 \qquad (73\text{-a})$$

$$G'(a|x';y|y') = 0 \qquad (73\text{-b})$$

$$G'(x|x';0|y') = 0 \qquad (73\text{-c})$$

$$G'(x|x';b|y') = 0 \qquad (73\text{-d})$$

For the discrete Green's function on a mesh with step sizes h_R and k_S Eq(72) takes the form (Ref 4:315)

$$\frac{\partial^2 G_N'(x|x';y|y')}{\partial x^2} + \frac{\partial^2 G_N'(x|x';y|y')}{\partial y^2}$$

$$= h_R^{-1} k_S^{-1} \delta(x-x') \delta(y-y') \tag{74}$$

The derivative terms in Eq(74) can be replaced by central difference quotients as (Ref 5:6)

$$\frac{\partial^2 G_N'(x|x';y|y')}{\partial x^2} \tag{75-a}$$

$$= \frac{G_N'(x+h_R|x';y|y') - 2G_N'(x|x';y|y') + G_N'(x-h_R|x';y|y')}{h_R^2}$$

$$\frac{\partial^2 G_N'(x|x';y|y')}{\partial x^2} \tag{75-b}$$

$$= \frac{G_N'(x|x';y+k_S|y') - 2G_N'(x|x';y|y') + G_N'(x|x';y-k_S|y')}{k_S^2}$$

Substituting Eqs(75) into Eq(74) and multiplying by $h_R^2 k_S^2$ yields

$$k_S^2 G_N'(x+h_R|x';y|y') - 2(h_R^2+k_S^2) G_N'(x|x';y|y')$$

$$+k_S^2 G_N'(x-h_R|x';y|y') + h_R^2 G_N'(x|x';y+k_S|y')$$

$$+h_R^2 G_N'(x|x';y-k_S|y') = h_R k_S \delta(x-x') \delta(y-y') \tag{76}$$

Applying Eq(76) together with the boundary conditions (Eq(73)) at each of the N interior nodes of the mesh results in a set of N^2 simultaneous equations expressed in matrix form as

$$\bar{\bar{C}}'\bar{\bar{G}}_N' = h_R k_S \bar{\bar{I}}_N \tag{77}$$

where

$\bar{\bar{C}}'$ = The coefficient matrix

$$\bar{\bar{G}}_N' = \begin{bmatrix} G_N(x_1|x_1;y_1|y_1) & \cdots & G_N(x_1|x_N;y_1|y_N) \\ G_N(x_2|x_1;y_2|y_1) & \cdots & G_N(x_2|x_N;y_2|y_N) \\ \cdots & \cdots & \cdots \\ G_N(x_N|x_1;y_N|y_1) & \cdots & G_N(x_N|x_N;y_N|y_N) \end{bmatrix} \tag{78}$$

$\bar{\bar{I}}_N$ = The identity matrix of order N

The solution of Eq(77) yields the approximate discrete Green's function matrix (Eq(78)). The solution to Poisson's equation in two dimensions then follows as in Eq(64).

Computer Analysis

The one dimensional computer program was adapted to handle the two-dimensional Galerkin and finite difference approximations on a rectangular body. The equations programmed were Eq(65-a) and Eq(64). The resulting computer program is on file with the AFIT Department of Physics.

Problem Set. A series of four problems were analyzed as the number of interior nodes of the mesh was increased.

The areas of consideration were: convergence rate, computer time, and the overall solution accuracy. The four problems analyzed were:

$$\text{I)} \quad L'T'(x,y) = 10 \tag{79-a}$$

$$\text{II)} \quad L'T'(x,y) = x^2 \tag{79-b}$$

$$\text{III)} \quad L'T'(x,y) = x^2 + y^2 \tag{79-c}$$

$$\text{IV)} \quad L'T'(x,y) = x^2 + y^2 + x \tag{79-d}$$

The Dirichlet boundary conditions for all four problems were taken as

$$T'(0,y) = 0 \tag{80-a}$$

$$T'(2,y) = 0 \tag{80-b}$$

$$T'(x,0) = 0 \tag{80-c}$$

$$T'(x,2) = 0 \tag{80-d}$$

The analysis was performed at the following four points for all cases:

Point 1 $(x = 0.\overline{66},\ y = 0.\overline{66})$

Point 2 $(x = 1.\overline{33},\ y = 0.\overline{66})$

Point 3 $(x = 0.\overline{66},\ y = 1.\overline{33})$

Point 4 $(x = 1.\overline{33},\ y = 1.\overline{33})$

Exact Solutions. The analytical solutions to Eqs(79) were found by integrating Eq(46) with the respective inhomogeneity terms. The analytical solutions are presented as a single function where the inhomogeneity is taken as

$$g'(x,y) = \alpha x^2 + \beta y^2 + \gamma x + \delta y + \epsilon \qquad (81)$$

where α, β, γ, δ, and ϵ are constants. The solution is then

$$T'(x,y) = -\frac{4}{\pi^2 ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left[\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]\right]\right.$$

$$\left[\alpha\left\{\left(\frac{2a^3}{m^3\pi^3}-\frac{a^3}{m\pi}\right)(-1)^m-\frac{2a^3}{m^3\pi^3}\right\}\left(\frac{b}{n\pi}\right)(1-(-1)^n)\right.$$

$$+\beta\left\{\left(\frac{2b^3}{n^3\pi^3}-\frac{b^3}{n\pi}\right)(-1)^n-\frac{2b^3}{n^3\pi^3}\right\}\left(\frac{a}{m\pi}\right)(1-(-1)^m)$$

$$-\gamma\left(\frac{a^2b(-1)^m}{mn\pi^2}\right)(1-(-1)^n)-\delta\left(\frac{b^2a(-1)^n}{mn\pi^2}\right)(1-(-1)^m)$$

$$\left.\left.+\epsilon\left(\frac{ab}{mn\pi^2}\right)((-1)^m-1)((-1)^n-1)\right]\right] \qquad (82)$$

The values of the above solution, for the problems in Eq(79), at the points of interest are listed in Table II. Each listed value was obtained using 100 summation terms.

Average Error. The average percent error for the two-dimensional problem is defined as

Table II

Exact Solution Values for the Two-Dimensional Problem Set at Points of Interest

Problem No.	Point 1	Point 2	Point 3	Point 4
I'	-2.41384	-2.41384	-2.41384	-2.41384
II'	-0.20168	-0.36879	-0.20168	-0.36879
III'	-0.40336	-0.57047	-0.57047	-0.73759
IV'	-0.60297	-0.85364	-0.77008	-1.02075

$$\langle E_N' \rangle = \frac{100}{4} \sum_{i=1}^{4} \left\{ \frac{|T_N'(\text{point } i) - T'(\text{point } i)|}{T'(\text{point } i)} \right\} \quad (83)$$

where

T_N' = The approximate solution under consideration

Convergence Rate. The rate at which each of the two approximate solutions (Eqs(65-a) and (64)) converged toward the exact values listed in Table II was examined for problems I'-IV'. The average percent error in the approximate solutions is plotted vs the number of interior nodes in the mesh as Figures 11-14. Appendix B contains the actual values of the approximations at the points of interest for each of the four problems.

In each of the four cases the Galerkin approximations

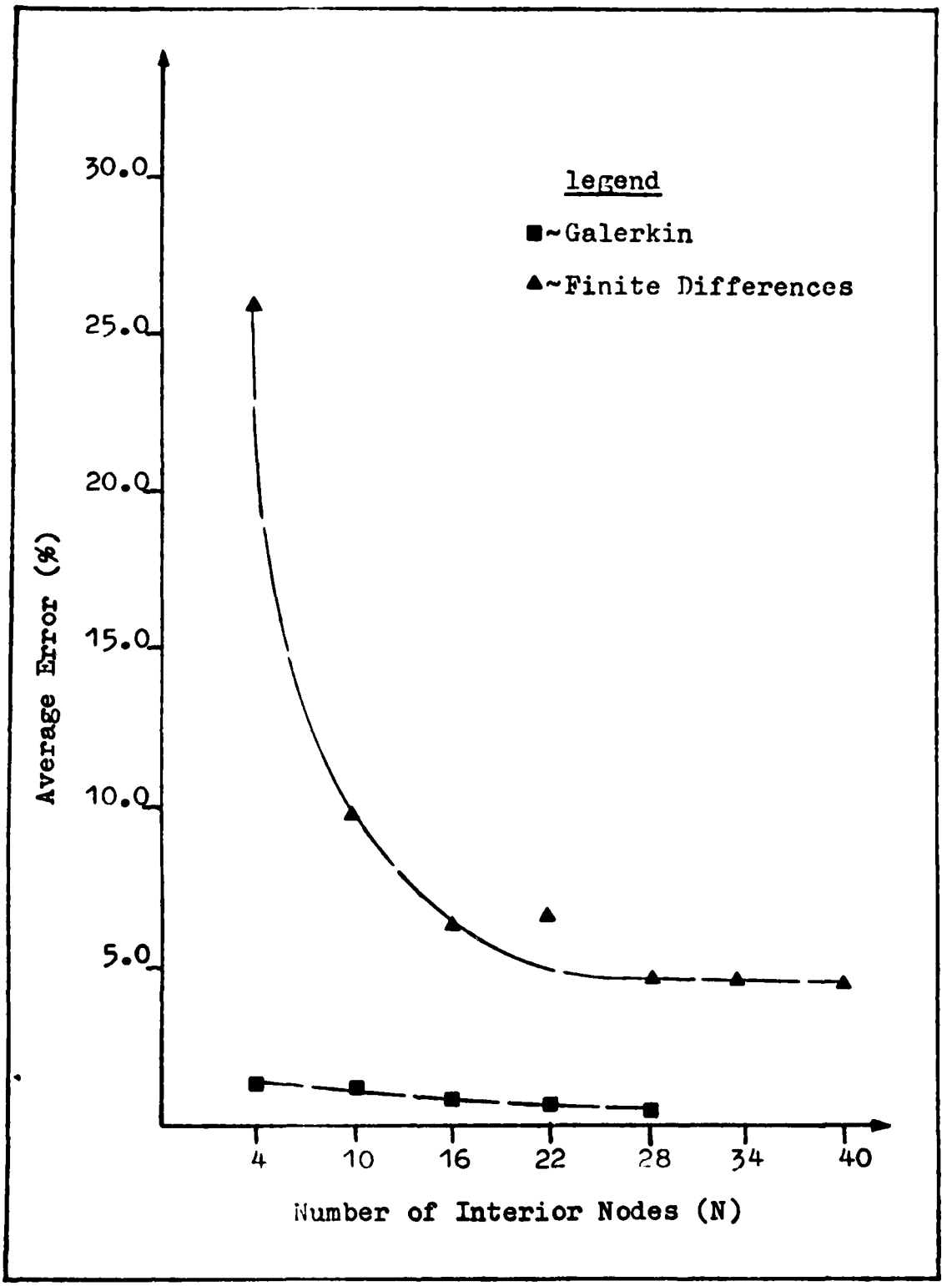


Figure 11. Average Error vs Node Number for g'(x,y) = 10

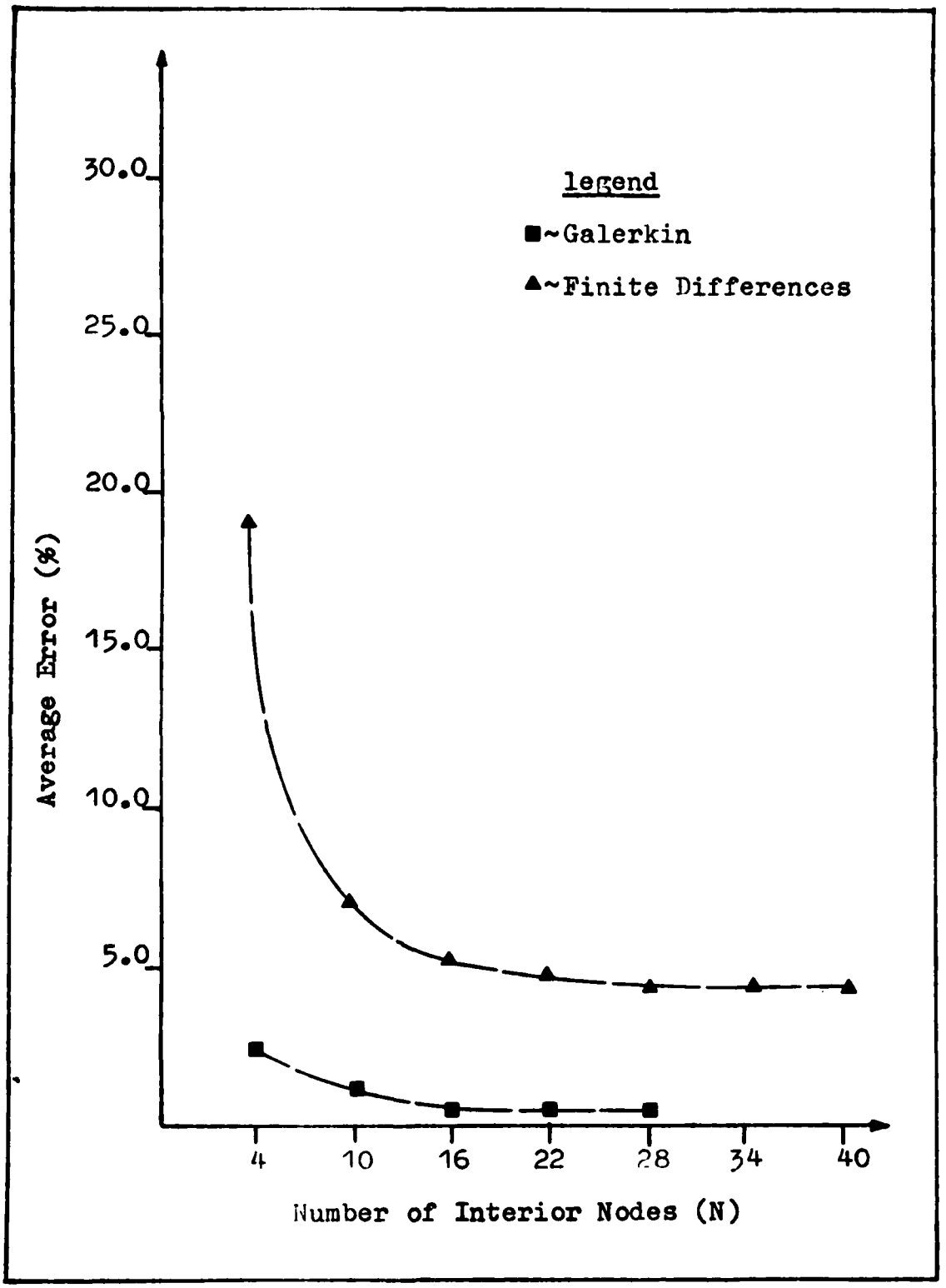


Figure 12. Average Error vs Node Number for $g'(x,y) = x^2$

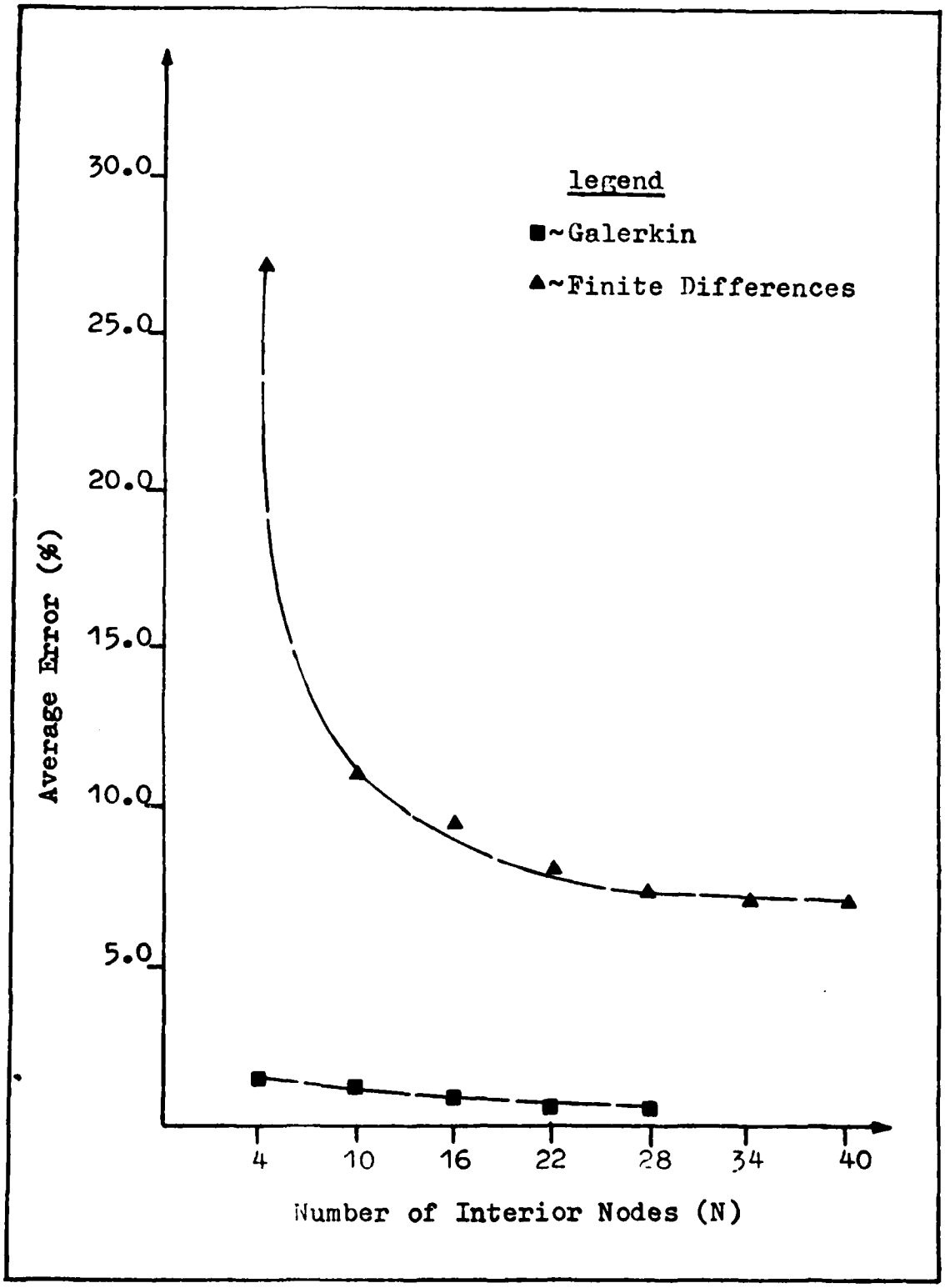


Figure 13. Average Error vs Node Number for $g'(x,y) = x^2 + y^2$

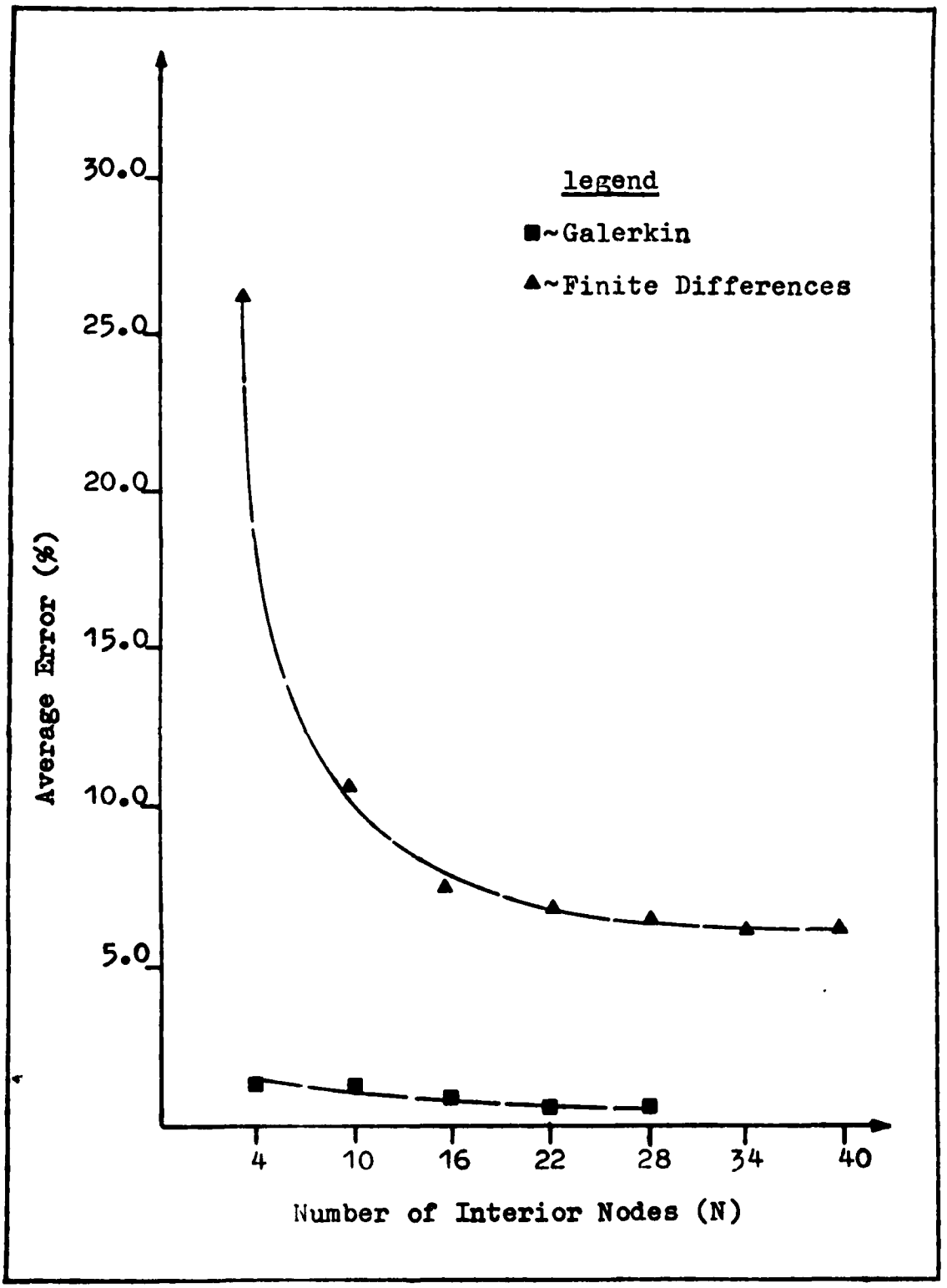


Figure 14. Average Error vs Node Number for $g'(x,y) = x^2 + y^2 + x$

yielded the best initial solution. Initial average errors of between one and two percent were noted. In contrast, the initial average errors for the finite difference approximations ranged from 18 to 27 percent.

The Galerkin approximations yielded average errors below one percent for 16 or more interior nodes. The finite difference approximations never had an average error smaller than four percent. Both methods converged at the same rate for 10 or more interior nodes.

The above trends are inconsistent with the one-dimensional cases where the finite difference approximations were consistently better than the Galerkin approximations. The general trends should be the same for both one and two dimensions. The two-dimensional finite difference approximations seem to converge rapidly but toward a value well above the exact solutions. It is suspected that there is a minor error in the programming of the finite difference method which could explain this discrepancy.

No oscillations comparable to the one-dimensional collocation cases were noted except for a slight increase in average error for the Galerkin in problem I with 22 nodes and for the finite differences in problem II with 22 nodes.

Computer Analysis Time. The quantity used as a measure of the computer analysis time was the actual execution time of the program. The time taken by compilation, input, and output was almost constant for all computer runs. The execution time in seconds is plotted in Figures 15-18 vs

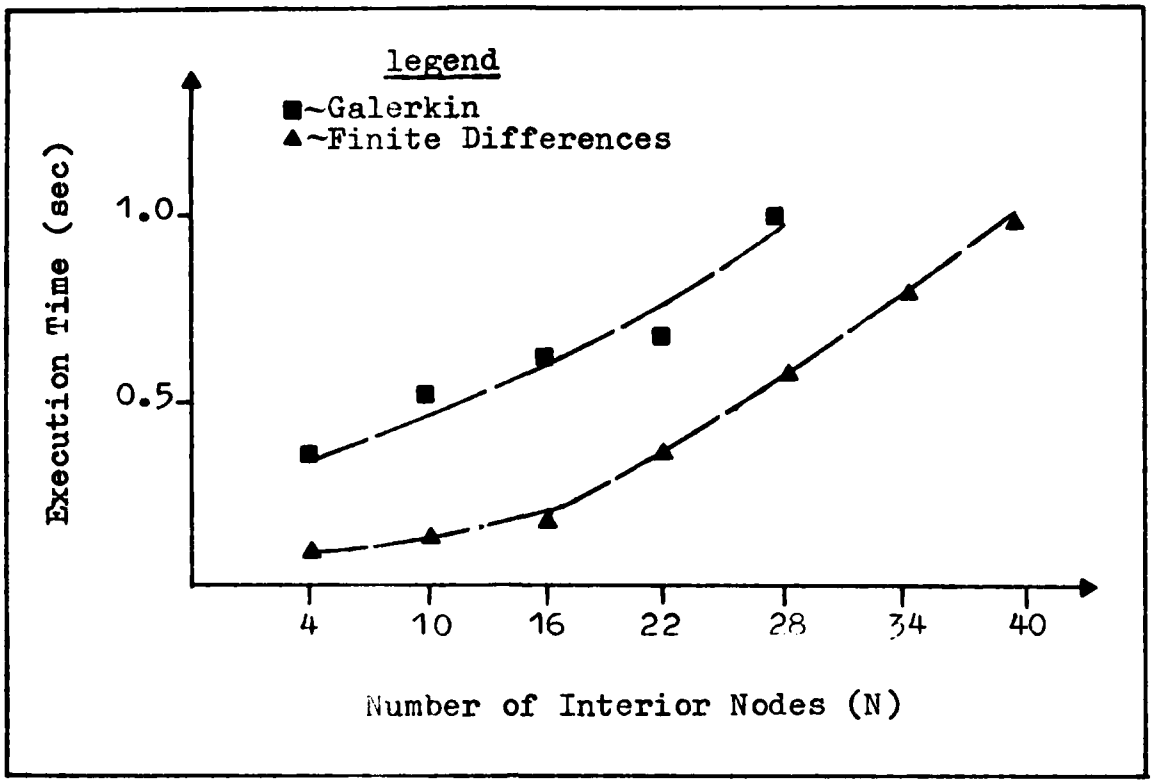


Figure 15. Execution Time vs Node Number for $g'(x,y) = 10$

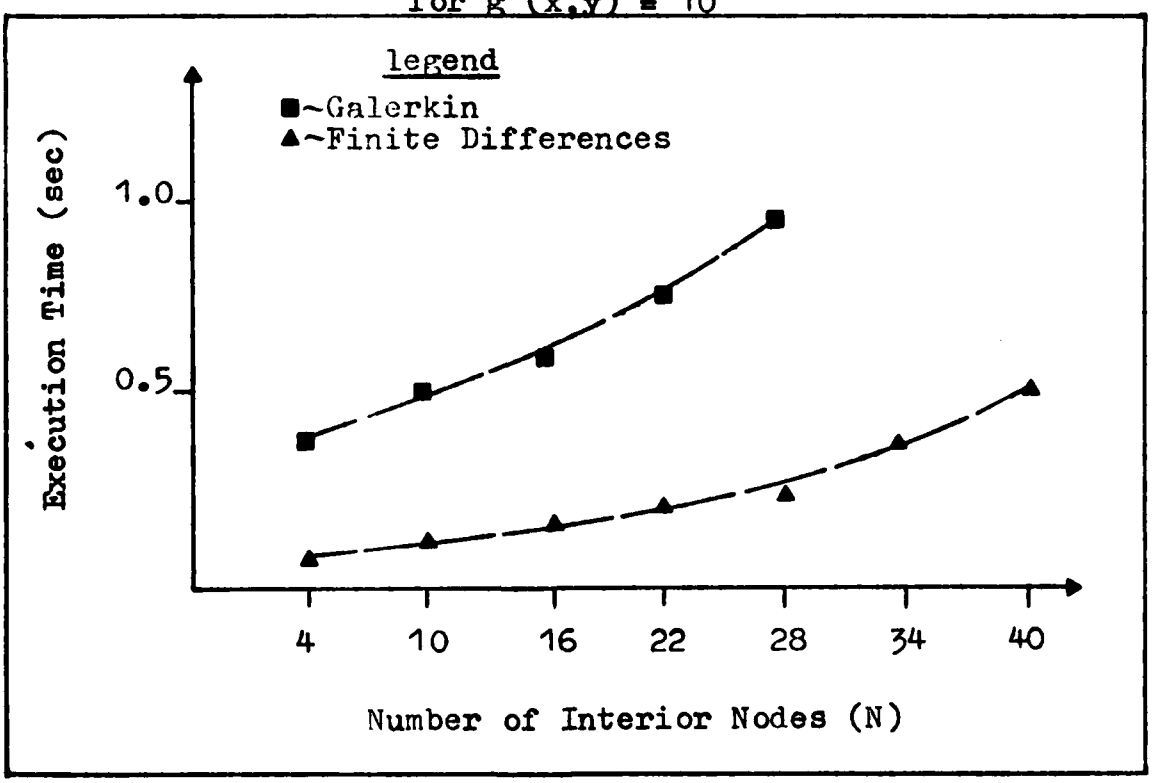


Figure 16. Execution Time vs Node Number for $g'(x,y) = x^2$

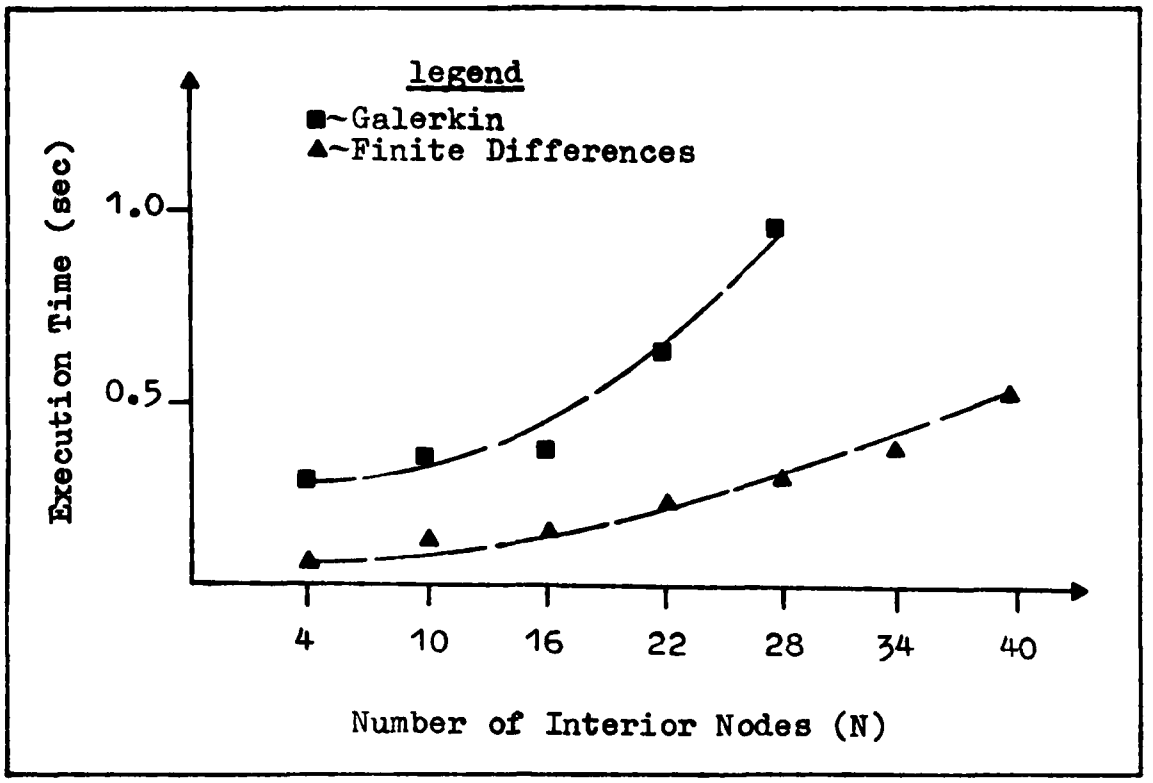


Figure 17. Execution Time vs Node Number for $g'(x,y) = x^2 + y^2$

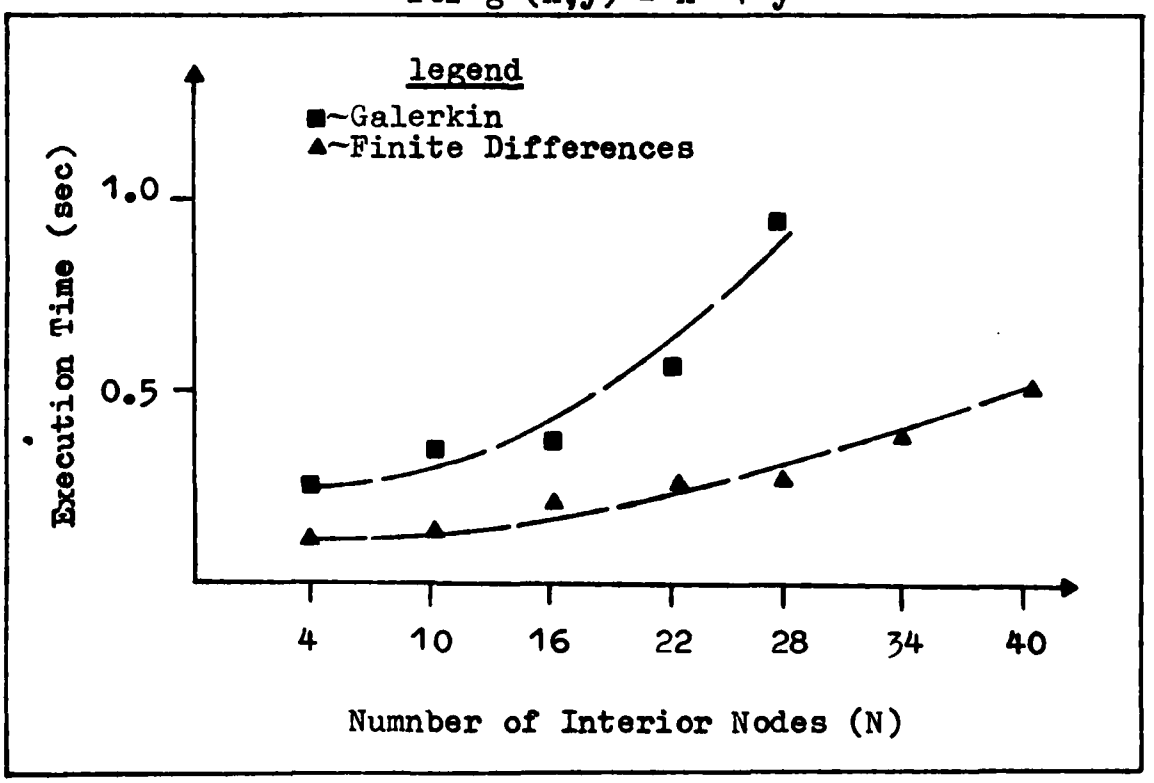


Figure 18. Execution Time vs Node Number for $g'(x,y) = x^2 + y^2 + x$

the number of interior nodes in the mesh.

In all cases the Galerkin approximations took more time to execute than did the finite difference approximations. This is due to the additional matrix multiplication required in the Galerkin method.

The trends in Figures 15-18 indicate that for more than 28 interior nodes the Galerkin approximations will require more than one second of computer time while the finite difference approximations will require only about 0.5 seconds. This is an important trend when a high-accuracy, many-term approximation is desired.

Overall Solution Accuracy. The convergence trends indicate that both the Galerkin and finite difference approximations are slowly converging and will not reach an average error less than 0.1 percent unless a large number (>60) of interior nodes are used. The Galerkin method will yield better accuracy.

Conclusions

From the analyses performed it seems that neither of the two approximation methods will yield very accurate answers when few interior nodes are used. At least 16 nodes are required before the Galerkin method will yield average errors less than one percent. The finite difference approximations never reach that level of accuracy.

For a large number of interior nodes (between 50 and 100) the two methods are about equal. The accuracy limitations of the finite difference approximations trade-off

against the large execution time requirements of the Galerkin approximations. If the suspected programming error in the finite difference method is located and corrected then, as in the one-dimensional case, the method should be the best choice to use.

IV. Conclusions and Recommendations

Conclusions

Three points may be made regarding the use of the method of weighted residuals to determine approximations to the discrete Green's function. First, the Galerkin and collocation methods both yield analogs which are as useful as the discrete Green's function itself. They allowed the solution of Poisson's equation in one and two dimensions (only the one-dimensional case was verified for collocation) with four different inhomogeneity terms. The accuracy of these approximate solutions was limited to the accuracies of the methods themselves. That is, the solutions were the same as those obtained if the methods were applied to each problem without incorporating Green's functions or analogs. The only major limitation to the method of Galerkin is that the inhomogeneity matrix must be recalculated for each respective inhomogeneity term.

The second point is the apparent sensitivity of the weighted residual approximations to the choice of the trial functions. The one-dimensional collocation approximations displayed a damped oscillation not present in the finite difference approximations. Although this may be a result of the round-off error (indicated by the failure of the direct matrix inversion routine's accuracy test) in the matrix inversion, the fact that convergence was still fairly rapid tends to say it is not. The more likely cause is an inappropriate choice of the trial functions for the problems

examined.

Finally, the method of weighted residuals requires more computer time than does the method of finite differences. This does not become important until large numbers of interior nodes are used. In the two-dimensional analysis it was seen that a large number (100) of interior nodes was necessary to insure an average error of less than 0.1 percent. In these cases the execution time analysis indicated a rapid growth in the time required to the extent that it may become prohibitive in terms of computer resources. A large part of this problem might be eliminated by the streamlining of the existing computer program and using more efficient routines where applicable.

Recommendations

Three areas from this study warrant further work. The first is a study of the two-dimensional method of collocation. This should be the initial task before any new work is undertaken. This will allow a full comparison between the current analysis and any new additions. The methods of least-squares and subsectional basing (Ref 2:11-14) might also be examined.

The second area is the analysis of different trial functions to see if they affect the accuracy of the resulting solutions. A possible choice would be the eigenfunctions of the Laplacian operator (Ref 6:241-245). Particular attention should be paid to the oscillation in the collocation approximations.

The last area for further work is the aforementioned streamlining of the existing computer program. Included in this would also be the location and correction of any errors already present. The matrix inversion program is the routine considered the most time consuming. It can probably be replaced by a more efficient routine while still maintaining the desired degree of accuracy. If a lower accuracy routine is incorporated attention should be paid to any induced round-off errors.

Bibliography

1. Ozisik, M. N. Boundary Value Problems of Heat Conduction. Scranton, Pennsylvania: International Textbook Company, 1968.

2. Harrington, R. F. Field Computation by Moment Methods. New York: The Macmillan Company, 1968.

3. Stackgold, I. Boundary Value Problems of Mathematical Physics. Vol. II. New York: The Macmillan Company, 1967.

4. Forsythe, G. E. and W. R. Wasow. Finite Difference Methods for Partial Differential Equations. New York: John Wiley and Sons, Inc., 1962.

5. Smith, G. D. Numerical Solution of Partial Differential Equations. London, New York, N.Y.: Oxford University Press, 1965.

6. Babuska, I., M. Prager and E. Vitasek. Numerical Processes in Differential Equations. London: John Wiley and Sons, Ltd., 1966.

7. IMSL Library Reference Manual, Edition 8. IMSL, Inc. Houston, Texas. 1980.

8. Gallof, S. A Numerical Investigation and Utilization of Green's Functions. Unpublished thesis. Wright-Patterson Air Force Base, Ohio: Air Force Institute of Technology, June 1970.

Appendix A

Numerical Approximations for the One-Dimensional Problem Set

This Appendix contains the values of the Galerkin, collocation, and finite difference approximations to Eqs(40). The values are the output of the computer analysis and are tabulated according to inhomogeneity and the number of interior nodes. The average percent error is also listed for each set of values.

Note: The notation (*) indicates that for the given number of interior nodes the direct matrix inversion program used indicated that its accuracy test had failed.

The notation (") indicates a repeated value from the line above it.

Table III

Galerkin Approximations for $g(x) = 10$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-3.29218	-4.27984	14.81469
5	-4.17238	-4.38043	3.78079
8	-4.33062	-4.40286	1.74825
11	-4.35919	-4.37570	1.73239
14	-4.34807	-4.39567	1.63283
17	-4.35842	-4.35400	1.98518
20	-4.35830	-4.41338	1.31850

Table IV

Galerkin Approximations for $g(x) = x^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.35117	-0.61454	9.85043
5	-0.40985	-0.62125	2.45861
8	-0.42040	-0.62274	1.10697
11	-0.42200	-0.62087	1.06953
14	-0.42096	-0.62235	1.07273
17	-0.42157	-0.62019	1.17412
20	-0.42347	-0.62817	0.31427

Table V

Galerkin Approximations for $g(x) = x^2 + 1$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.68038	-1.04252	12.28890
5	-0.82708	-1.05229	3.42479
8	-0.85346	-1.06303	1.41104
11	-0.85791	-1.05844	1.37050
14	-0.85577	-1.06191	1.33099
17	-0.85741	-1.05559	1.53233
20	-0.85930	-1.06951	0.77352

Table VI

Galerkin Approximations for $g(x) = x^2 + x + 1$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.99863	-1.52538	11.83411
5	-1.20401	-1.54885	2.98185
8	-1.24093	-1.55408	1.35820
11	-1.24716	-1.55766	1.56347
14	-1.24413	-1.55252	1.28185
17	-1.24642	-1.54370	1.47352
20	-1.24919	-1.56343	0.73341

Table VII

Collocation Approximations for $g(x) = 10$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-2.96296	-2.96296	33.33333
5	*-4.19151	*-4.69738	5.69104
8	*-3.97101	*-3.97101	10.65219
11	*-4.32441	*-4.56448	2.70079
14	*-4.17154	*-4.17153	6.14048
17	*-4.36832	*-4.52071	1.71439
20	*-4.25132	*-4.23779	4.49742

Table VIII

Collocation Approximations for $g(x) = x^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.32922	-0.52675	19.43228
5	*-0.41112	*-0.64238	3.31822
8	*-0.39642	*-0.59395	6.20983
11	*-0.41998	*-0.63352	1.57490
14	*-0.40979	*-0.60732	3.57911
17	*-0.42291	*-0.63060	0.99919
20	*-0.41511	*-0.61152	2.62186

Table IX

Collocation Approximations for $g(x) = x^2 + 1$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.62551	-0.82305	25.68956
5	*-0.83027	*-1.11211	4.38594
8	*-0.79352	*-0.99105	8.20995
11	*-0.85242	*-1.08996	2.08141
14	*-0.82694	*-1.02447	4.73287
17	*-0.85974	*-1.08267	1.32123
20	*-0.84024	*-1.03530	3.46453

Table X

Collocation Approximations for $g(x) = x^2 + x + 1$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.92181	-1.21811	24.68903
5	*-1.20847	*-1.62280	6.54321
8	*-1.15702	*-1.45332	7.88995
11	*-1.23948	*-1.59179	2.00020
14	*-1.20381	*-1.50011	4.54813
17	*-1.24973	*-1.58158	1.26941
20	*-1.22243	*-1.51511	3.33400

Table XI

Finite Difference Approximations for $g(x) = 10$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-4.44444	-4.44444	0.00000
5	"	"	"
8	"	"	"
11	"	"	"
14	"	"	"
17	"	"	"
20	"	"	"

Table XII

Finite Difference Approximations for $g(x) = x^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.39506	-0.59259	6.47743
5	-0.41976	-0.61728	1.61819
8	-0.42433	-0.62186	0.71818
11	-0.42593	-0.62436	0.40336
14	-0.42667	-0.62420	0.25776
17	-0.42707	-0.62460	0.17905
20	-0.42731	-0.62484	0.13183

Table XIII

Finite Difference Approximations for $g(x) = x^2 + 1$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-0.83951	-1.03704	3.42506
5	-0.86420	-1.06173	0.85626
8	-0.86877	-1.06630	0.38079
11	-0.87037	-1.06790	0.21433
14	-0.87111	-1.06864	0.13734
17	-0.87151	-1.06904	0.09572
20	-0.87176	-1.06929	0.06971

Table XIV

Finite Difference Approximations for $g(x) = x^2 + x + 1$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$	Average Error (%)
2	-1.23456	-1.53086	2.35191
5	-1.25926	-1.55556	0.78504
8	-1.26383	-1.56013	0.26140
11	-1.26543	-1.56173	0.14713
14	-1.26617	-1.56247	0.09428
17	-1.26658	-1.56287	0.05585
20	-1.26682	-1.56311	0.04817

Appendix B

Numerical Approximations for the Two-Dimensional Problem Set

This Appendix contains the values of the Galerkin and finite difference approximations to Eqs(79). The values are the output of the computer analysis and are tabulated according to inhomogeneity and the number of interior nodes. The average percent error is also listed for each set of values.

Note: The notation (**) indicates that for the given number of interior nodes the direct matrix inversion program indicated that the matrix $\overline{\overline{I}}'$ was algorithmically singular. The resulting solution values were discarded as being invalid.

Table XV

Galerkin Approximations for $g'(x,y) = 10$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-2.40852	-2.46212	-2.46212	-2.52302	2.18594
10	-2.45819	-2.43633	-2.43633	-2.38168	1.25826
16	-2.41038	-2.43064	-2.43064	-2.40911	0.43282
22	-2.39171	-2.41861	-2.41861	-2.39383	0.53525
28	-2.39441	-2.42412	-2.42393	-2.40121	0.54301
34	**	**	**	**	———
40	**	**	**	**	———

Table XVI

Galerkin Approximations for $g'(x,y) = x^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-0.19932	-0.37005	-0.20008	-0.38251	1.50636
10	-0.19916	-0.37799	-0.20014	-0.36458	1.41233
16	-0.19877	-0.37342	-0.20255	-0.36642	0.94309
22	-0.19927	-0.37225	-0.20148	-0.36654	0.71061
28	-0.19957	-0.37219	-0.20155	-0.36652	0.66203
34	**	**	**	**	———
40	**	**	**	**	———

Table XVII

Galerkin Approximations for $g'(x,y) = x^2 + y^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-0.39863	-0.57013	-0.57013	-0.76502	1.25049
10	-0.39831	-0.57813	-0.57813	-0.72917	1.27232
16	-0.39755	-0.57597	-0.57597	-0.73285	0.97297
22	-0.39854	-0.57373	-0.57372	-0.73308	0.73910
28	-0.39905	-0.57364	-0.57361	-0.73304	0.70008
34	**	**	**	**	———
40	**	**	**	**	———

Table XVIII

Galerkin Approximations for $g'(x,y) = x^2 + y^2 + x$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-0.59706	-0.85163	-0.77124	-1.05504	1.18138
10	-0.59666	-0.86734	-0.77552	-1.01021	1.09759
16	-0.59570	-0.86139	-0.77607	-1.01482	0.86809
22	-0.59694	-0.85854	-0.77330	-1.01512	0.63594
28	-0.59761	-0.85842	-0.77328	-1.01507	0.64254
34	**	**	**	**	———
40	**	**	**	**	———

Table XIX

Finite Difference Approximations for $g'(x,y) = 10$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-2.05128	-1.62393	-2.13675	-2.05128	18.56088
10	-2.26123	-2.18765	-2.27367	-2.25634	7.03412
16	-2.29383	-2.26968	-2.29771	-2.29182	5.20250
22	-2.30374	-2.29305	-2.30543	-2.30279	4.66425
28	-2.30789	-2.30225	-2.30877	-2.30737	4.44396
34	-2.30998	-2.30665	-2.31050	-2.30967	4.33500
40	-2.31116	-2.30905	-2.31150	-2.31097	4.27410

Table XX

Finite Difference Approximations for $g'(x,y) = x^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-0.14815	-0.23457	-0.16049	-0.29630	25.75409
10	-0.18346	-0.32431	-0.18583	-0.33749	9.36034
16	-0.19016	-0.34152	-0.19096	-0.34612	6.14224
22	-0.19232	-0.34689	-0.19269	-0.32858	6.48503
28	-0.19326	-0.34912	-0.19345	-0.35025	4.65414
34	-0.19373	-0.35022	-0.19385	-0.35089	4.42834
40	-0.19401	-0.35083	-0.19408	-0.35127	4.29801

Table XXI

Finite Difference Approximations for $g'(x,y) = x^2 + y^2$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-0.29630	-0.32099	-0.46914	-0.59259	26.92389
10	-0.35657	-0.48655	-0.51500	-0.66410	11.49946
16	-0.36691	-0.51467	-0.52383	-0.67811	8.76445
22	-0.37013	-0.52310	-0.52676	-0.68267	7.91249
28	-0.37149	-0.52650	-0.52805	-0.68463	7.55623
34	-0.37217	-0.52816	-0.52871	-0.68564	7.37818
40	-0.37256	-0.52907	-0.52911	-0.68621	7.27728

Table XXII

Finite Difference Approximations for $g'(x,y) = x^2 + y^2 + x$

Number of Interior Nodes	Point 1 $x = 0.\overline{66}$ $y = 0.\overline{66}$	Point 2 $x = 1.\overline{33}$ $y = 0.\overline{66}$	Point 3 $x = 0.\overline{66}$ $y = 1.\overline{33}$	Point 4 $x = 1.\overline{33}$ $y = 1.\overline{33}$	Average Error (%)
4	-0.46154	-0.51045	-0.64435	-0.83191	24.62142
10	-0.54301	-0.74068	-0.70309	-0.92738	10.25579
16	-0.55685	-0.77945	-0.71430	-0.94595	7.72779
22	-0.56112	-0.79102	-0.71799	-0.95196	6.94492
28	-0.56291	-0.79568	-0.71959	-0.95453	6.61934
34	-0.56381	-0.79793	-0.72042	-0.95584	6.45710
40	-0.56431	-0.79917	-0.72091	-0.95659	6.36578

Vita

Randolph E. Clapp was born on 17 September 1958 in Smithtown, New York, the son of Ralph E. and Grace M. Clapp. After graduating from Smithtown High School East, in Smithtown, New York, in 1976, he accepted an appointment to the United States Air Force Academy. In May, 1980, he was graduated and commisioned as a second lieutenant in the United States Air Force. His first assignment was to the Rome Air Development Center detachment located at Hanscom Air Force Base, Massachusetts. He was a member of the Antennas and RF Components branch working in the area of adaptive nulling for the protection of phased array antennas from jamming. In May, 1982, he was assigned to the Air Force Institute of Technology, WPAFB, Ohio.

Permanent address: 4 Rutgers Street
Smithtown, New York 11787

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFIT/GEP/PH/83D-3	2. GOVT ACCESSION NO. AD-A138374	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) INVESTIGATION OF THE NUMERICAL METHOD OF MOMENTS FOR DIGITAL COMPUTER DETERMINATION OF GREEN'S FUNCTIONS		5. TYPE OF REPORT & PERIOD COVERED MS Thesis 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Randolph E. Clapp 1/Lt		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Air Force Institute of Technology (AFIT-EN) Wright-Patterson AFB, Ohio 45433		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Materials Laboratory (AFWAL/MLBM) Wright-Patterson AFB, Ohio 45433		12. REPORT DATE October, 1983 13. NUMBER OF PAGES 76
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Approved for public release: IAW AFR 190-17.

7 Feb 84

LYNN E. WOLAVER
Dean for Research and Professional Development
Air Force Institute of Technology (AIC)
Wright-Patterson AFB OH 45433

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Green's Functions
Method of Weighted Residuals
Method of Galerkin
Method of Collocation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This study was to determine the feasibility of using the method of weighted residuals to obtain approximations to the discrete Green's function or an analog to it. The methods of Galerkin, collocation, and finite differences were programmed in Fortran IV. The resulting program was used to generate the approximate functions for the one and two-dimensional Poisson's equation. The two-dimensional case was restricted to the

Block 19:

Method of Finite Differences

Block 20:

methods of Galerkin and finite differences on a rectangular body. The approximate Green's functions and analogs were applied to a series of inhomogeneity terms to obtain the approximate solutions. The average percent error of the approximate solutions is reported as the number of interior nodes of the mesh was increased. The areas of consideration were: the rate of convergence of the approximate solutions toward the analytical solution, the computer-time required to execute the methods, and the accuracy of the approximate solutions. The results of this study indicate that the Green's functions and analogs obtained are valid approximations to the discrete Green's function, with the restrictions that additional calculations may be required in the Galerkin approximations and excessive computer-time may occur for high-accuracy approximations. The finite difference approximations were determined to be the better method to use.

END

FILMED

4-84

DTIC